

Besov-Type Spaces with Variable Smoothness and Integrability

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Abstract In this article, the authors introduce Besov-type spaces with variable smoothness and integrability. The authors then establish their characterizations, respectively, in terms of φ -transforms in the sense of Frazier and Jawerth, smooth atoms or Peetre maximal functions, as well as a Sobolev-type embedding. As an application of their atomic characterization, the authors obtain a trace theorem of these variable Besov-type spaces.

1 Introduction

Spaces of variable integrability, also known as variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, can be traced back to Orlicz [55, 56], and studied by Musielak [45] and Nakano [49, 50], but the modern development started with the articles [31] of Kováčik and Rákosník as well as [8] of Cruz-Uribe and [13] of Diening. The variable Lebesgue spaces have already widely used in the study of harmonic analysis; see, for example, [10, 9, 11, 14, 15, 48, 26]. Apart from theoretical considerations, such function spaces have interesting applications in fluid dynamics [1, 57], image processing [7], partial differential equations and variational calculus [2, 20, 25, 54, 58].

In recent years, function spaces with variable exponents attract many attentions, especially based on classical Besov and Triebel-Lizorkin spaces (see Triebel's monographs [60, 61, 62] for the history of these two spaces). When Leopold [33, 34, 35, 36] and Leopold and Schrohe [37] studied pseudo-differential operators, they introduced related Besov spaces with variable smoothness, $B_{p,p}^{s(\cdot)}(\mathbb{R}^n)$, which were further generalized to the case that $q \neq p$, including $B_{p,q}^{s(\cdot)}(\mathbb{R}^n)$ and $F_{p,q}^{s(\cdot)}(\mathbb{R}^n)$, by Besov [4, 5, 6]. Along a different line of study, Xu [66, 67] studied Besov spaces $B_{p(\cdot),q}^s(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F_{p(\cdot),q}^s(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ but fixed q and s . As was well known from the trace theorem (see, for example, [22, Theorem 11.1]) and Sobolev-type embeddings (see, for example, [60, Theorem 2.7.1]) of classical function spaces, the smoothness and the integrability often interact each other. However, the unification of both trace theorems and Sobolev-type embeddings does not hold true on function spaces with only one variable index; for example, the trace space of Sobolev space $W^{k,p(\cdot)}$ is no longer a space of the same type (see [15]). Thus, function spaces with full ranges of variable smoothness and variable integrability are needed.

The concept of function spaces with variable smoothness and variable integrability was firstly mixed up by Diening, Hästö and Roudenko in [16], they introduced Triebel-Lizorkin spaces with variable exponents $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ and proved a trace theorem as follows:

$$\mathrm{Tr} F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) = F_{p(\cdot,0),p(\cdot,0)}^{s(\cdot,0)-1/p(\cdot,0)}(\mathbb{R}^{n-1}),$$

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(see [16, Theorem 3.13]), which shows that these spaces behaved nicely with respect to the trace operator. Subsequently, Vybíral [65] established Sobolev-Jawerth embeddings of these spaces. On the other hand, Almeida and Hästö [3] introduced the Besov space with variable smoothness and integrability $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$, which makes a further step in completing the unification process of function spaces with variable smoothness and integrability. Later, Drihem [17] established the atomic characterization of $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ and Noi et al. [51, 52, 53] also studied the space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ including the boundedness of trace and extension operators, duality and complex interpolation. Here we point out that vector-valued convolution inequalities developed in [3, Lemma 4.7] and [16, Theorem 3.2] supply well remedy for the absence of the Fefferman-Stein vector-valued inequality for the mixed Lebesgue sequence spaces $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ and $L^{p(\cdot)}(\ell^{q(\cdot)}(\mathbb{R}^n))$, respectively, in studying Besov spaces and Triebel-Lizorkin spaces with variable smoothness and integrability.

More generally, 2-microlocal Besov and Triebel-Lizorkin spaces with variable, $B_{p(\cdot),q(\cdot)}^{w(\cdot)}(\mathbb{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{w(\cdot)}(\mathbb{R}^n)$, were introduced by Kempka [27, 28] and provided a unified approach that cover the classical Besov and Triebel-Lizorkin spaces as well as versions of variable smoothness and integrability. Afterwards, Kempka and Vybíral [29] characterized these spaces by local means and ball means of differences. The trace spaces of 2-microlocal type spaces were studied very recently by Moura et al. [44] and Gonçalves et al. [24].

On the other hand, Besov-type spaces $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and their homogeneous counterparts for all admissible parameters were introduced in [68, 69, 78] in order to clarify the relations among Besov spaces, Triebel-Lizorkin spaces and Q space (see [12, 19]). Various properties and equivalent characterizations of Besov-type and Triebel-Lizorkin-type spaces, including smoothness atomic, molecular or wavelet decompositions, characterizations, respectively, via differences, oscillations, Peetre maximal functions, Lusin area functions or g_λ^* functions, have already been established in [18, 42, 70, 71, 72, 73, 74, 77]. Moreover, these function spaces, including some of their special cases related to Q spaces, have been used to study the existence and the regularity of solutions of some partial differential equations such as (fractional) Navier-Stokes equations; see, for example, [38, 39, 40, 41, 63, 79, 64]. Based on $F_{p,q}^{s,\tau}(\mathbb{R}^n)$, we introduced the Triebel-Lizorkin-type space with variable exponent $F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ in [76] with a measurable function ϕ on \mathbb{R}_+^{n+1} and obtained a related trace theorem ([76, Theorem 4.1]).

In this article, based on Besov-type spaces $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and variable Besov spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$, we are aimed to introduce another more generalized scale of function spaces with variable smoothness $s(\cdot)$, variable integrability $p(\cdot)$ and $q(\cdot)$, and a measurable function ϕ on \mathbb{R}_+^{n+1} , denoted by $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, which covers both Besov spaces with variable smoothness and integrability and Besov-type spaces. We then establish their φ -transform characterization in the sense of Frazier and Jawerth. We also characterize these spaces by smooth atoms or Peetre maximal functions in this article and give some basic properties and Sobolev-type embeddings. As applications, we prove a trace theorem of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ and obtain several equivalent norms of these spaces.

This article is organized as follows.

In Section 2, we first give some conventions and notation such as semimodular spaces, variable and mixed Lebesgue-sequence spaces, and also introduce variable Besov-type spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. We point out that the function spaces studied in this article fit into the framework of so-called semimodular spaces. At the end of this section, we point out that, in general, the scale of Besov-type spaces with variable smoothness and integrability and the scale of Musielak-Orlicz Besov-type spaces in [75] do not cover each other (see Remark 2.15 below).

Section 3 is devoted to the φ -transform characterization of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ in the sense of Frazier and Jawerth [22], which is then applied to show that $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ is well defined. This is different

from [3, Theorem 5.5], in which the space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ was proved to be well defined via the Calderón reproducing formula. We point out that the method used in this article is originally from Frazier and Jawerth [22], which is smartly modified in this article, via a subtle decomposition of dyadic cubes, so that it is suitable to the present setting (see Theorem 3.3 and Corollary 3.5 below). Observe that the *r-trick lemma* from [16, Lemma A.6] (see also Lemma 3.9 below) plays a key role in establishing a convolutional estimate so that we can use the convolutional inequality from [3, Lemma 4.7] (see also Lemma 3.12 below) to obtain the desired conclusion.

In Section 4, by making full use of the *r-trick lemma* from [16, Lemma A.6] again, we mainly give out the Sobolev-type embedding property of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ (see Proposition 4.2 and Theorem 4.3 below). Some other basic embeddings and properties of the spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ are also presented.

In Section 5, we first characterize the space $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ via Peetre maximal functions (see Theorem 5.1 below). A key step to obtain this is to establish a technical lemma (see Lemma 5.4 below), which indicates that the Peetre maximal function can be controlled, via semimodulars, by the approximation to the identity in a suitable way. Applying Theorem 5.1, we further obtain two equivalent characterizations of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ (see Theorem 5.5 below). Finally, in this section, by applying a Hardy-type inequality from [18, Lemma 3.11] (see also Lemma 5.12 below) and the Sobolev-type embedding theorem obtained in Section 4, together with some ideas from the proof of Lemma 5.4, we establish the smooth atomic characterization of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ (see Theorem 5.9 below).

In the last section, Section 6, as an application of the smoothness atomic characterization obtained in Theorem 5.9, we prove a trace theorem for $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ (see Theorem 6.1 below), which partly extends the corresponding one obtained in [44, Theorem 3.4] and also [51, Theorem 5.1(1)]. The key point for this is to prove that the trace space of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ is independent of the n -th coordinate of variable exponents $p(\cdot)$, $q(\cdot)$ and $s(\cdot)$ (see Corollary 6.6 and Lemma 6.7 below).

2 Preliminary

Throughout the article, we denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. The *symbols* $A \lesssim B$ means $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For all $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. For all $k := (k_1, \dots, k_n) \in \mathbb{Z}^n$, let $|k| := |k_1| + \dots + |k_n|$. Let $\mathbb{Z}_+ := \{0, 1, \dots\}$, $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{K} := \mathbb{R}$ or \mathbb{C} . Let $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times [0, \infty)$. If E is a subset of \mathbb{R}^n , we denote by χ_E its *characteristic function* and $\tilde{\chi}_E := |E|^{-1/2}\chi_E$. For all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, denote by $Q(x, r)$ the cube centered at x with side length r , whose sides parallel axes of coordinate. For all cube $Q \subset \mathbb{R}^n$, we denote its *center* by c_Q and its *side length* by $\ell(Q)$ and, for $a \in (0, \infty)$, we denote by aQ the *cube* concentric with Q having the side length with $a\ell(Q)$.

2.1 Modular spaces

In this subsection, we recall some conventions and notions about (semi)modular spaces, and state some basic results. For an exposition of these concepts, we refer to the monograph [15, Chapters 1-3]. The function spaces studied in this article fit into the framework of so-called semimodular spaces. In what follows, let X be a vector space over \mathbb{K} .

Definition 2.1. A function $\varrho : X \rightarrow [0, \infty]$ is called a *semimodular* on X if it satisfies:

- (i) $\varrho(0) = 0$ and, for all $f \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| = 1$, $\varrho(\lambda f) = \varrho(f)$;
- (ii) If $\varrho(\lambda f) = 0$ for all $\lambda \in (0, \infty)$, then $f = 0$;

(iii) ρ is *quasiconvex*, namely, there exists $A \in [1, \infty)$ such that, for all $f, g \in X$,

$$\rho(\theta f + (1 - \theta)g) \leq A[\theta \rho(f) + (1 - \theta)\rho(g)];$$

(iv) $\lambda \mapsto \rho(\lambda f)$ is left continuous on $[0, \infty)$ for every $f \in X$, namely, $\lim_{\lambda < 1, \lambda \rightarrow 1} \rho(\lambda f) = \rho(f)$.

A semimodular ρ is called a *modular* if it satisfies that $\rho(f) = 0$ implies $f = 0$, and is called *continuous* if, for every $f \in X$, the mapping $\lambda \mapsto \rho(\lambda f)$ is continuous on $[0, \infty)$, namely, $\lim_{\lambda \rightarrow 1} \rho(\lambda f) = \rho(f)$.

Definition 2.2. Let ρ be a (semi)modular on X . Then

$$X_\rho := \{f \in X : \exists \lambda \in (0, \infty) \text{ such that } \rho(\lambda f) < \infty\}$$

is called a *(semi)modular space* with the norm

$$\|f\|_\rho := \inf \{\lambda \in (0, \infty) : \rho(f/\lambda) \leq 1\}.$$

The following Lemma 2.3 is just [15, Lemma 2.1.14].

Lemma 2.3. Let ρ be a semimodular on X . Then $\|f\|_\rho \leq 1$ if and only if $\rho(f) \leq 1$; moreover, if ρ is continuous, then $\|f\|_\rho < 1$ if and only if $\rho(f) < 1$, as well as $\|f\|_\rho = 1$ if and only if $\rho(f) = 1$.

Remark 2.4. When dealing with some complicated quasi-norms defined via variable exponents, we are often converted to dealing with the corresponding semimodulars by Lemma 2.3; see Remarks 2.5 and 2.9(i) below.

2.2 Spaces of variable integrability

Here, we recall some definitions and notation for the space with variable integrability. For a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty]$, let

$$p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

The set of *variable exponents* in this article, denoted by $\mathcal{P}(\mathbb{R}^n)$, is the set of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty]$ satisfying $p_- \in (0, \infty]$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the function $\rho_{p(x)}$ by setting, for all $t \in [0, \infty)$,

$$\rho_{p(x)}(t) := \begin{cases} t^{p(x)}, & \text{if } p(x) \in (0, \infty), \\ 0, & \text{if } p(x) = \infty \text{ and } t \in [0, 1], \\ \infty, & \text{if } p(x) = \infty \text{ and } t \in (1, \infty). \end{cases}$$

The *variable exponent modular* of a measurable function f on \mathbb{R}^n is defined by

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|) dx.$$

Remark 2.5. Let $p \in \mathcal{P}(\mathbb{R}^n)$ satisfy $p_- \in [1, \infty]$. Then $\rho_{p(\cdot)}$ is a semimodular (see [15, Definition 3.2.1]), which, together with Lemma 2.3, implies that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$ if and only if $\rho_{p(\cdot)}(f) \leq 1$. Moreover, for all $p \in \mathcal{P}(\mathbb{R}^n)$, $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$ if and only if $\rho_{p(\cdot)}(f) \leq 1$.

Definition 2.6. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of \mathbb{R}^n . Then the *variable exponent Lebesgue space* $L^{p(\cdot)}(E)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}(E)} := \inf \{\lambda \in (0, \infty) : \rho_{p(\cdot)}(f\chi_E/\lambda) \leq 1\} < \infty.$$

Remark 2.7. Let $p \in \mathcal{P}(\mathbb{R}^n)$.

(i) If $p_- \in [1, \infty]$, then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space (see [15, Theorem 3.2.7]). In particular, for all $\lambda \in \mathbb{C}$, $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and, for all $f, g \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f + g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

(ii) (**The Hölder inequality**) Assume that $1 < p_- \leq p_+ < \infty$. It was proved in [15, Lemma 3.2.20] that, if $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p^*(\cdot)}(\mathbb{R}^n)$, then $fg \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p^*(\cdot)}(\mathbb{R}^n)},$$

where $p^*(x) := \frac{p(x)}{p(x)-1}$ for all $x \in \mathbb{R}^n$, C is a positive constant depending on p_- or p_+ , but independent of f and g .

(iii) If $p_+ \in (0, 1]$, then it is easy to see that, for all nonnegative functions $f, g \in L^{p(\cdot)}(\mathbb{R}^n)$, the following reverse Minkowski inequality holds true:

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f + g\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Definition 2.8. Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of \mathbb{R}^n . Then the *mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)}(E))$ is defined to be the set of all sequences $\{f_v\}_{v \in \mathbb{N}}$ of functions in $L^{p(\cdot)}(E)$ such that

$$\|\{f_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(E))} := \inf \left\{ \lambda \in (0, \infty) : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v \chi_E / \lambda\}_{v \in \mathbb{N}}) \leq 1 \right\} < \infty,$$

where, for all sequences $\{g_v\}_{v \in \mathbb{N}}$ of measurable functions,

$$(2.1) \quad \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{g_v\}_{v \in \mathbb{N}}) := \sum_{v \in \mathbb{N}} \inf \left\{ \mu_v \in (0, \infty) : \varrho_{p(\cdot)}\left(g_v / \mu_v^{1/q(\cdot)}\right) \leq 1 \right\}$$

with the convention $\lambda^{1/\infty} = 1$ for all $\lambda \in (0, \infty)$.

Remark 2.9. Let $p, q \in \mathcal{P}(\mathbb{R}^n)$.

(i) The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ was introduced by Almeida and Hästö [3]. Moreover, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ is a semimodular (see [3, Proposition 3.5]), which, together with Lemma 2.3, implies that $\|f\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} \leq 1$ if and only if $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(f) \leq 1$.

(ii) If $q_+ \in (0, \infty)$, then, for all measurable functions g on \mathbb{R}^n , it holds true that

$$\inf \left\{ \lambda \in (0, \infty) : \varrho_{p(\cdot)}\left(\frac{g}{\lambda^{1/q(\cdot)}}\right) \leq 1 \right\} = \left\| |g|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}.$$

(iii) Let $\{g_v\}_{v \in \mathbb{N}}$ be a sequence of functions in $L^{p(\cdot)}(\mathbb{R}^n)$. If, for all $v \in \{2, 3, \dots\}$, $g_v \equiv 0$, then

$$\|\{g_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} = \|g_1\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

(see [3, Example 3.4]).

(iv) If $p, q \in \mathcal{P}(\mathbb{R}^n)$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}$ is a quasi-norm on $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ (see [3, Theorem 3.8]); if either $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ or q is a constant, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}$ is a norm (see [3, Theorem 3.6]); if either $p(x) \geq 1$ and $q \in [1, \infty)$ is a constant almost everywhere or $1 \leq q(x) \leq p(x) \leq \infty$ for almost every $x \in \mathbb{R}^n$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}$ is also a norm (see [30, Theorem 1]).

(v) By [3, Proposition 3.3], we know that, if $q \in (0, \infty]$ is constant, then

$$\|\{g_v\}_{v \in \mathbb{N}}\|_{\ell^q(L^{p(\cdot)}(\mathbb{R}^n))} = \left\| \left\{ \|g_v\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{v \in \mathbb{N}} \right\|_{\ell^q}.$$

A measurable function $g \in \mathcal{P}(\mathbb{R}^n)$ is said to satisfy the *locally log-Hölder continuous condition*, denoted by $g \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$, if there exists a positive constant $C_{\log}(g)$ such that, for all $x, y \in \mathbb{R}^n$,

$$(2.2) \quad |g(x) - g(y)| \leq \frac{C_{\log}(g)}{\log(e + 1/|x - y|)},$$

and g is said to satisfy the *globally log-Hölder continuous condition*, denoted by $g \in C^{\log}(\mathbb{R}^n)$, if $g \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$ and there exist positive constants C_{∞} and g_{∞} such that, for all $x \in \mathbb{R}^n$,

$$(2.3) \quad |g(x) - g_{\infty}| \leq \frac{C_{\infty}}{\log(e + |x|)}.$$

Remark 2.10. (i) Let $p \in C^{\log}(\mathbb{R}^n)$. Then, it was proved in [15, Lemma 4.6.3] that, for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and every nonnegative, radially decreasing function $g \in L^1(\mathbb{R}^n)$,

$$\|f * g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

where C is a positive constant independent of f and g .

(ii) Let $p \in \mathcal{P}(\mathbb{R}^n)$. If $p_+ \in (0, \infty)$, then $p \in C^{\log}(\mathbb{R}^n)$ if and only if $1/p \in C^{\log}(\mathbb{R}^n)$. If p satisfies (2.3), then $p_{\infty} = \lim_{|x| \rightarrow \infty} p(x)$.

(iii) If $q \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$ and $q_+ = \infty$, then, by (2.2), it is easy to see that $q(x) = \infty$ for all $x \in \mathbb{R}^n$. From this and Remark 2.9(v), we deduce that, in the case that $q_+ = \infty$, the mixed norm $\|\cdot\|_{\ell q(\cdot)(L^{p(\cdot)}(\mathbb{R}^n))}$ becomes the norm $\|\cdot\|_{\ell^{\infty}(L^{p(\cdot)}(\mathbb{R}^n))}$.

2.3 The Besov-type space $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$

Let $\mathcal{G}(\mathbb{R}_+^{n+1})$ be the set of all measurable functions $\phi : \mathbb{R}_+^{n+1} \rightarrow (0, \infty)$ having the following properties: there exist positive constants c_1 and c_2 such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$(2.4) \quad c_1^{-1} \phi(x, 2r) \leq \phi(x, r) \leq c_1 \phi(x, 2r)$$

and, for all $x, y \in \mathbb{R}^n$ and $r \in (0, \infty)$ with $|x - y| \leq r$,

$$(2.5) \quad c_2^{-1} \phi(y, r) \leq \phi(x, r) \leq c_2 \phi(y, r).$$

Remark 2.11. (i) We point out that (2.4) and (2.5) are called the *doubling condition* and the *compatibility condition*, respectively, which have been used by Nakai [46, 47] and Nakai and Sawano [48] when they studied generalized Campanato spaces.

(ii) There are several examples of ϕ that satisfy (2.4) and (2.5); see [76, Remark 1.3].

In what follows, for $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$ and all cubes $Q := Q(x, r) \subset \mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r \in (0, \infty)$, define $\phi(Q) := \phi(Q(x, r)) := \phi(x, r)$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on \mathbb{R}^n and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual space. A pair of functions, (φ, Φ) , is said to be *admissible* if $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$(2.6) \quad \text{supp } \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \text{ and } |\widehat{\varphi}(\xi)| \geq c > 0 \text{ when } 3/5 \leq |\xi| \leq 5/3$$

and

$$(2.7) \quad \text{supp } \widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \text{ and } |\widehat{\Phi}(\xi)| \geq c > 0 \text{ when } |\xi| \leq 5/3,$$

where $\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx$ for all $\xi \in \mathbb{R}^n$ and c is a positive constant independent of $\xi \in \mathbb{R}^n$. For all $j \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we put $\varphi_j(x) := 2^{jn} \varphi(2^j x)$ and $\widetilde{\varphi}(x) := \overline{\varphi(-x)}$. For $j \in \mathbb{Z}$

and $k \in \mathbb{Z}^n$, denote by Q_{jk} the *dyadic cube* $2^{-j}([0, 1]^n + k)$, $x_{Q_{jk}} := 2^{-j}k$ its *lower left corner* and $\ell(Q_{jk})$ its *side length*. Let $\mathcal{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, $\mathcal{Q}^* := \{Q \in \mathcal{Q} : \ell(Q) \leq 1\}$ and $j_Q := -\log_2 \ell(Q)$ for all $Q \in \mathcal{Q}$.

Now we introduce the Besov-type space with variable smoothness and integrability.

Definition 2.12. Let (φ, Φ) be a pair of admissible functions on \mathbb{R}^n . Let $p, q \in C^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Then the *Besov-type space with variable smoothness and integrability*, $B^{s(\cdot), \phi}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s(\cdot), \phi}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} |\varphi_j * f| \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,$$

where the supremum is taken over all dyadic cubes P in \mathbb{R}^n .

Remark 2.13. Let p, q, s and ϕ be as in Definition 2.12.

(i) If $\phi(Q) = 1$ for all cubes Q of \mathbb{R}^n , then $B^{s(\cdot), \phi}_{p(\cdot), q(\cdot)}(\mathbb{R}^n) = B^{s(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$, where $B^{s(\cdot)}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ denotes the *Besov space with variable smoothness and integrability* introduced in [3].

(ii) If p, q, s are constant exponents and $\phi(Q) := |Q|^\tau$ with $\tau \in [0, \infty)$ for all cubes Q of \mathbb{R}^n , then $B^{s(\cdot), \phi}_{p(\cdot), q(\cdot)}(\mathbb{R}^n) = B^{s, \tau}_{p, q}(\mathbb{R}^n)$, where $B^{s, \tau}_{p, q}(\mathbb{R}^n)$ denotes the *Besov-type space* introduced in [78].

(iii) By Remark 2.10(iii), we see that, when $q_+ = \infty$,

$$\|f\|_{B^{s(\cdot), \phi}_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} = \|f\|_{B^{s(\cdot), \phi}_{p(\cdot), \infty}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \geq (j_P \vee 0)} \left\| 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(P)}.$$

(iv) If q, s are constants and ϕ is as in (ii), then $B^{s(\cdot), \phi}_{p(\cdot), q(\cdot)}(\mathbb{R}^n) = B^{s, \tau}_{p(\cdot), q}(\mathbb{R}^n)$, which was investigated in [43].

We end this section by comparing Besov-type spaces with variable smoothness and integrability in this article with Musielak-Orlicz Besov-type spaces in [75] and show that, in general, these two scales of Besov-type spaces do not cover each other.

To recall the definition of Musielak-Orlicz Besov-type spaces, we need some notions on Musielak-Orlicz functions. A function $\varphi : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$ is called a *Musielak-Orlicz function* if the function $\varphi(x, \cdot) : [0, \infty) \rightarrow [0, \infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$, namely, for any given $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is nondecreasing, $\varphi(x, 0) = 0$, $\varphi(x, t) \in (0, \infty)$ for all $t \in (0, \infty)$ and $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$, and $\varphi(\cdot, t)$ is a Lebesgue measurable function for all $t \in [0, \infty)$. A Musielak-Orlicz function φ is said to be of *uniformly upper* (resp. *lower*) *type* p for some $p \in [0, \infty)$ if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in [1, \infty)$ (resp. $s \in [0, 1]$), $\varphi(x, st) \leq C s^p \varphi(x, t)$ (see [32]). Let

$$i(\varphi) := \sup\{p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p\}$$

and

$$I(\varphi) := \inf\{p \in (0, \infty) : \varphi \text{ is of uniformly upper type } p\}.$$

The function $\varphi(\cdot, t)$ is said to satisfy the *uniformly Muckenhoupt condition for some* $r \in [1, \infty)$, denoted by $\varphi \in \mathbb{A}_r(\mathbb{R}^n)$, if, when $r \in (1, \infty)$,

$$\sup_{t \in (0, \infty)} \sup_{\text{balls } B \subset \mathbb{R}^n} \frac{1}{|B|^r} \int_B \varphi(x, t) dx \left\{ \int_B [\varphi(y, t)]^{-r'/r} dy \right\}^{r/r'} < \infty,$$

where $1/r + 1/r' = 1$, or, when $r = 1$,

$$\sup_{t \in (0, \infty)} \sup_{\text{balls } B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x, t) dx \left\{ \text{ess sup}_{y \in B} [\varphi(y, t)]^{-1} \right\} < \infty.$$

Let $\mathbb{A}_\infty(\mathbb{R}^n) := \cup_{r \in [1, \infty)} \mathbb{A}_r(\mathbb{R}^n)$.

The *Musielak-Orlicz space* $L^\varphi(\mathbb{R}^n)$ is defined as the set of all measurable functions f on \mathbb{R}^n such that

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda) dx \leq 1 \right\} < \infty.$$

Let $\mathcal{S}_\infty(\mathbb{R}^n)$ be the *space* of all Schwartz functions h satisfying that, for all multi-indices $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n$, $\int_{\mathbb{R}^n} h(x) x^\gamma dx = 0$ and let $\mathcal{S}'_\infty(\mathbb{R}^n)$ be its *topological dual space*. Now we recall the definition of Musielak-Orlicz Besov-type spaces from [75] as follows.

Definition 2.14. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $q \in (0, \infty]$ and ψ be a Schwartz function satisfying $\text{supp } \widehat{\psi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$ and $|\widehat{\psi}(\xi)| \geq C > 0$ if $3/5 \leq |\xi| \leq 5/3$ for some positive constant C independent of $\xi \in \mathbb{R}^n$. For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\psi_j(x) := 2^{jn} \psi(2^j x)$. Assume that, for $j \in \{1, 2\}$, φ_j is a Musielak-Orlicz function with $0 < i(\varphi_j) \leq I(\varphi_j) < \infty$ and $\varphi_j \in \mathbb{A}_\infty(\mathbb{R}^n)$. Then the *Musielak-Orlicz Besov-type space* $\dot{B}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}_{\varphi_1, \varphi_2, q}^{s, \tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\|\chi_P\|_{L^{\varphi_1}(\mathbb{R}^n)}} \left\| \left\{ \sum_{j=j_P}^{\infty} (2^{js} |\psi_j * f|)^q \right\}^{1/q} \right\|_{L^{\varphi_2}(\mathbb{R}^n)} < \infty$$

with suitable modification made when $q = \infty$, where the supremum is taken over all dyadic cubes P of \mathbb{R}^n .

Remark 2.15. (i) Observe that, if $\varphi(x, t) := t^{p(x)}$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, then $L^\varphi(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$.

(ii) Let φ_1 be as in Definition 2.14. If $\phi(P) := \|\chi_P\|_{L^{\varphi_1}(\mathbb{R}^n)}$ for all cubes $P \subset \mathbb{R}^n$, then, by [80, Lemma 2.6] and [76, Remark 1.3(iv)], we see that ϕ satisfies (2.4) and (2.5).

(iii) The scale of Besov-type spaces with variable smoothness and integrability can not be covered by the scale of Musielak-Orlicz Besov-type spaces. Indeed, by [75, Remark 2.23(iii)], we find that there exists some function $p(\cdot)$ satisfying conditions in Definition 2.12, but $t^{p(\cdot)}$ is not a Musielak-Orlicz function as in Definition 2.14.

(iv) Also, the scale of Besov-type spaces with variable smoothness and integrability can not cover the scale of Musielak-Orlicz Besov-type spaces, since a Musielak-Orlicz function $\varphi(x, t)$ may not be written as $\varphi(x, t) := t^{p(x)}$ for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$ with some variable exponent $p(\cdot)$ as in Definition 2.12 (see, for example, the Musielak-Orlicz function φ as in [75, (1.5)]).

3 The φ -transform characterization

The purpose of this section is to show that $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ is independent of the choice of admissible function pairs (φ, Φ) . To this end, we first introduce the sequence space $b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ with respect to $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ and then establish its φ -transform characterization in the sense of Frazier and Jawerth [22].

Definition 3.1. Let p, q, s and ϕ be as in Definition 2.12. Then the *sequence space* $b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ is defined to be the set of all sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ such that

$$\|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q) = 2^{-j}}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell q(\cdot)(L^{p(\cdot)}(P))} < \infty,$$

where the supremum is taken over all dyadic cubes P in \mathbb{R}^n .

Remark 3.2. Let $\mathcal{D}_0(\mathbb{R}^n) := \{Q \subset \mathbb{R}^n : Q \text{ is a cube and } \ell(Q) = 2^{-j_0} \text{ for some } j_0 \in \mathbb{Z}\}$. Then the supremum in Definitions 2.12 and 3.1 can be equivalently taken over all cubes in $\mathcal{D}_0(\mathbb{R}^n)$, the details being omitted.

Let (φ, Φ) be a pair of admissible functions. Then $(\tilde{\varphi}, \tilde{\Phi})$ is also a pair of admissible functions, where $\tilde{\varphi}(\cdot) := \varphi(-\cdot)$ and $\tilde{\Phi}(\cdot) := \Phi(-\cdot)$. Moreover, by [22, pp. 130-131] or [23, Lemma (6.9)], there exist Schwartz functions ψ and Ψ satisfying (2.6) and (2.7), respectively, such that, for all $\xi \in \mathbb{R}^n$,

$$(3.1) \quad \widehat{\Phi}(\xi) \widehat{\Psi}(\xi) + \sum_{j=1}^{\infty} \widehat{\varphi}(2^{-j}\xi) \widehat{\psi}(2^{-j}\xi) = 1.$$

Recall that the φ -transform S_φ is defined to be the mapping taking each $f \in \mathcal{S}'(\mathbb{R}^n)$ to the sequence $S_\varphi(f) := \{(S_\varphi f)_Q\}_{Q \in \mathcal{Q}^*}$, where $(S_\varphi f)_Q := |Q|^{1/2} \varphi_{j_Q} * f(x_Q)$ with φ_0 replaced by Φ ; the *inverse* φ -transform T_ψ is defined to be the mapping taking a sequence $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ to

$$(3.2) \quad T_\psi t := \sum_{Q \in \mathcal{Q}^*, \ell(Q)=1} t_Q \Psi_Q + \sum_{Q \in \mathcal{Q}^*, \ell(Q)<1} t_Q \psi_Q;$$

see, for example, [78, p. 31].

Now we state the following φ -transform characterization for $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, which is the main result of this section.

Theorem 3.3. *Let p, q, s and ϕ be as in Definition 2.12 and φ, ψ, Φ and Ψ as in (3.1). Then operators $S_\varphi : B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) \rightarrow b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ and $T_\psi : b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) \rightarrow B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$.*

Remark 3.4. (i) The conclusion of Theorem 3.3 is new even when $\phi \equiv 1$.

(ii) If p, q, s and ϕ are as in Remark 2.13(ii), then Theorem 3.3 goes back to [78, Theorem 2.1].

(iii) T_ψ is well defined for all $t \in b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$; see Lemma 3.8 below.

From Theorem 3.3 and an argument similar to that used in [22, Remark 2.6], we immediately deduce the following conclusion, the details being omitted.

Corollary 3.5. *With all notation as in Definition 2.12, the space $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ is independent of the choice of the admissible function pairs (φ, Φ) .*

The remainder of this section is to prove Theorem 3.3. We begin with the following Lemmas 3.6 and 3.7, which are just [76, Lemma 2.5] and [76, Lemma 2.6], respectively.

Lemma 3.6. *Let $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$. Then there exist positive constants C and \tilde{C} such that, for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$, $\phi(Q_{jk}) \leq C 2^{j \log_2 c_1} (|k| + 1)^{2 \log_2 c_1}$ and, for all $Q \in \mathcal{Q}$ and $l \in \mathbb{Z}^n$,*

$$\frac{\phi(Q + l\ell(Q))}{\phi(Q)} \leq \tilde{C} (1 + |l|)^{2 \log_2 c_1},$$

where c_1 is as in (2.4).

Lemma 3.7. *Let $p \in C^{\log}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all dyadic cubes Q_{jk} with $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$,*

$$C^{-1} 2^{-\frac{n}{p_-} j} (1 + |k|)^{n(\frac{1}{p_+} - \frac{1}{p_-})} \leq \|\chi_{Q_{jk}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C 2^{-\frac{n}{p_+} j} (1 + |k|)^{n(\frac{1}{p_-} - \frac{1}{p_+})}.$$

In what follows, for all $h \in \mathcal{S}(\mathbb{R}^n)$ and $M \in \mathbb{Z}_+$, let

$$\|h\|_{\mathcal{S}_M(\mathbb{R}^n)} := \sup_{|\gamma| \leq M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma h(x)| (1 + |x|)^{n+M+\gamma}.$$

Lemma 3.8. *Let p, q, s and ϕ be as in Definition 2.12. Then, for all $t \in b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, $T_\psi t$ in (3.2) converges in $\mathcal{S}'(\mathbb{R}^n)$; moreover, $T_\psi : b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous.*

Proof. Observe that, by Remark 2.9(iii), we find that, for any $Q \in \mathcal{Q}^*$,

$$\begin{aligned} |t_Q| &\leq \left\| |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(Q)} \|\chi_Q\|_{L^{p(\cdot)}(Q)}^{-1} |Q|^{\frac{s_-}{n} + \frac{1}{2}} \\ &\leq \left\| \left\{ \sum_{\substack{\tilde{Q} \subset Q, \tilde{Q} \in \mathcal{Q}^* \\ \ell(\tilde{Q}) = 2^{-j}}} |\tilde{Q}|^{-\frac{s(\cdot)}{n}} |t_{\tilde{Q}}| \tilde{\chi}_{\tilde{Q}} \right\}_{j \geq (j_Q \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(Q))} \frac{|Q|^{\frac{s_-}{n} + \frac{1}{2}}}{\|\chi_Q\|_{L^{p(\cdot)}(Q)}} \\ &\leq \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \frac{\phi(Q)}{\|\chi_Q\|_{L^{p(\cdot)}(Q)}} |Q|^{\frac{s_-}{n} + \frac{1}{2}}. \end{aligned}$$

Then, by this and an argument similar to that used in the proof of [76, Lemma 2.4], we conclude that, for all $h \in \mathcal{S}(\mathbb{R}^n)$, $|\langle T_\psi t, h \rangle| \lesssim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \|h\|_{\mathcal{S}_M(\mathbb{R}^n)}$ with some large $M \in (0, \infty)$, which completes the proof of Lemma 3.8. \square

In what follows, for any $m \in (0, \infty)$ and $j \in \mathbb{Z}$, let, for all $x \in \mathbb{R}^n$, $\eta_{j,m}(x) := 2^{jn}(1 + 2^j|x|)^{-m}$. The following lemma is the so-called *r-trick lemma*, which is [16, Lemma A.6].

Lemma 3.9. *Let $r \in (0, \infty)$, $v \in \mathbb{Z}_+$ and $m \in (n, \infty)$. Then there exists a positive constant C , only depending on r, m and n , such that, for all $x \in \mathbb{R}^n$ and $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp } \hat{g} \subset \{\xi : |\xi| \leq 2^{v+1}\}$, $\sup_{z \in Q} |g(z)| \leq C [\eta_{v,m} * (|g|^r)(x)]^{\frac{1}{r}}$, where $Q \in \mathcal{Q}$ contains x and $\ell(Q) = 2^{-v}$.*

The following Lemma 3.10 is just [29, Lemma 19], which is a variant of [16, Lemma 6.1].

Lemma 3.10. *Let $s \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$ and $d \in [C_{\log}(s), \infty)$, where $C_{\log}(s)$ denotes the constant as in (2.2) with g replaced by s . Then, for all $x, y \in \mathbb{R}^n$ and $v \in \mathbb{N}$, $2^{vs(x)} \eta_{v,m+d}(x-y) \leq C 2^{vs(y)} \eta_{v,m}(x-y)$ with C being a positive constant independent of x, y and v ; moreover, for all nonnegative measurable functions f , it holds true that*

$$2^{vs(x)} \eta_{v,m+d} * f(x) \leq C \eta_{v,m} * (2^{vs(\cdot)} f)(x), \quad x \in \mathbb{R}^n.$$

Remark 3.11. Using the same notion as in Lemma 3.10, if $\lambda \in [2^{-v}, 2^{-v} + \theta]$ with $\theta \in [0, \infty)$, then, by an argument similar to that used in the proof of Lemma 3.10, we conclude that there exists a positive constant C such that, for all $x \in \mathbb{R}^n$,

$$\lambda^{-s(x)} \eta_{v,m+d} * f(x) \leq C \eta_{v,m} * (\lambda^{-s(\cdot)} f)(x).$$

It is well known that the boundedness of the Hardy-Littlewood maximal operator plays a key role in the study of the classical theory of function spaces. However, in the case of variable function spaces, such boundedness is usually absence. For example, the Hardy-Littlewood maximal operator is in general not bounded on the mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ (see [3, Example 4.1]). As a suitable substitute, a convolution with radical decreasing functions fits very well into this scheme. Indeed, we have the following Lemma 3.12, which is just [3, Lemma 4.7] (see also [29, Lemma 10]).

Lemma 3.12. *Let $p, q \in C^{\log}(\mathbb{R}^n)$ satisfy $p_-, q_- \in [1, \infty]$ and $m \in (n + C_{\log}(1/q), \infty)$, where $C_{\log}(1/q)$ denotes the constant as in (2.2) with g replaced by $1/q$. Then there exists a positive constant C such that, for all sequences $\{f_v\}_{v \in \mathbb{N}}$ of measurable functions,*

$$\|\{\eta_{v,m} * f_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} \leq C \|\{f_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}.$$

Remark 3.13. In Lemma 3.12, we require that $p_-, q_- \geq 1$. However, the following observation that, for all $r \in (0, \infty)$ and sequences $\{g_v\}_{v \in \mathbb{N}} \subset \ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$,

$$\|\{g_v\}_{v \in \mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} = \|\{|g_v|^r\}_{v \in \mathbb{N}}\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n))}^{\frac{1}{r}}$$

makes it possible to apply Lemma 3.12 even when $p_-, q_- \in (0, 1)$.

Lemma 3.14. *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$, $q_+ \in (0, \infty)$ and f be a measurable function on \mathbb{R}^n .*

- (i) *If $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, then $\| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}$.*
- (ii) *If $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} > 1$, then $\| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_+}$.*
- (iii) *If $\| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \geq 1$, then $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}^{1/q_-}$.*
- (iv) *If $\| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} < 1$, then $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}^{1/q_+}$.*

Proof. By similarity, we only prove (i) and (iii). Let $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Then, by Remark 2.5 and the fact that $\| \frac{f}{\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|_{L^{p(\cdot)}(\mathbb{R}^n)} = 1$, we see that $\varrho_{p(\cdot)}(f/\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}) \leq 1$. Thus, if $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$, then

$$\varrho_{p(\cdot)}\left(\frac{f}{[\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}]^{1/q(\cdot)}}\right) \leq \varrho_{p(\cdot)}\left(\frac{f}{[\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}]^{1/q_-}}\right) = \varrho_{p(\cdot)}\left(\frac{f}{\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right) \leq 1,$$

which implies that $\| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}$ and then completes the proof of (i).

For (iii), if $\| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \geq 1$, then, for all $\lambda > \| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}$,

$$\varrho_{p(\cdot)}\left(f/\lambda^{1/q_-}\right) \leq \varrho_{p(\cdot)}\left(f/\lambda^{1/q(\cdot)}\right) \leq 1,$$

which implies that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \lambda^{\frac{1}{q_-}}$. By this and the arbitrariness of $\lambda > \| |f|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}$, we conclude that (iii) holds true, which completes the proof of Lemma 3.14. \square

For a sequence $t = \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$, $r \in (0, \infty)$ and $\lambda \in (0, \infty)$, let $t_{r,\lambda}^* := \{(t_{r,\lambda}^*)_Q\}_{Q \in \mathcal{Q}^*}$, where, for all $Q \in \mathcal{Q}^*$,

$$(t_{r,\lambda}^*)_Q := \left\{ \sum_{R \in \mathcal{Q}^*, \ell(R)=\ell(Q)} \frac{|t_R|^r}{[1 + \{\ell(R)\}^{-1}|x_R - x_Q|]^\lambda} \right\}^{\frac{1}{r}}.$$

Lemma 3.15. *Let p, q, s and ϕ be as in Definition 2.12, $r \in (0, \min\{p_-, q_-\})$ and*

$$\lambda \in (2n + C_{\log}(s) + 2r \log_2 c_1, \infty),$$

where $C_{\log}(s)$ denotes the constant as in (2.2) with g replaced by s , and c_1 is as in (2.4). Then there exists a constant $C \in [1, \infty)$ such that, for all $t \in b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$,

$$(3.3) \quad \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \leq \|t_{r,\lambda}^*\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \leq C \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

Proof. To prove this lemma, it suffices to show the second inequality of (3.3) since the first one holds true obviously. We first claim that, for all $t \in b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)$, $\|t_{r, \lambda}^*\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)} \lesssim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)}$. Indeed, observe that, for all $r \in (0, \min\{p_-, q_-\})$, $Q \in \mathcal{Q}^*$ and $x \in Q$,

$$(t_{r, \lambda}^*)_Q \sim \left[\eta_{j_Q, \lambda} * \left(\sum_{R \in \mathcal{Q}^*, \ell(R)=2^{-j_Q}} |t_R|^r \chi_R \right) (x) \right]^{\frac{1}{r}}.$$

Thus, by Lemma 3.10, Remark 2.13(iv) and Lemma 3.12, we see that

$$\begin{aligned} \|t_{r, \lambda}^*\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)} &\lesssim \left\| \left\{ \eta_{j, \lambda - C_{\log}(s)} * \left(\left[2^{js(\cdot)} \sum_{R \in \mathcal{Q}^*, \ell(R)=2^{-j}} |t_R| \tilde{\chi}_R \right]^r \right) \right\}_{j \in \mathbb{Z}_+} \right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n))}^{\frac{1}{r}} \\ &\lesssim \left\| \left\{ 2^{js(\cdot)} \sum_{R \in \mathcal{Q}^*, \ell(R)=2^{-j}} |t_R| \tilde{\chi}_R \right\}_{j \in \mathbb{Z}_+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} \sim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)}, \end{aligned}$$

which proves the above claim.

For all $P \in \mathcal{Q}$ and $Q \in \mathcal{Q}^*$, let $v_Q^P := t_Q$ if $Q \subset 4P$ and $v_Q^P := 0$ otherwise, and let $u_Q^P := t_Q - v_Q^P$. Let $v^P := \{v_Q^P\}_{Q \in \mathcal{Q}^*}$ and $u^P := \{u_Q^P\}_{Q \in \mathcal{Q}^*}$. Then, we have

$$\begin{aligned} (3.4) \quad \|t_{r, \lambda}^*\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} &\leq \sup_{P \in \mathcal{Q}} \left\{ \frac{1}{\phi(P)} \left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s(\cdot)}{n}} |(v^P)_{r, \lambda}^*|_Q \tilde{\chi}_Q \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \right. \\ &\quad \left. + \frac{1}{\phi(P)} \left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s(\cdot)}{n}} |(u^P)_{r, \lambda}^*|_Q \tilde{\chi}_Q \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \right\} \\ &=: \sup_{P \in \mathcal{Q}} (I_{P,1} + I_{P,2}). \end{aligned}$$

By the above claim, (2.4) and Remark 3.2, we find that

$$\begin{aligned} (3.5) \quad I_{P,1} &\leq \frac{1}{\phi(P)} \|(v^P)_{r, \lambda}^*\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)} \lesssim \frac{1}{\phi(P)} \|v^P\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), 1}(\mathbb{R}^n)} \\ &\lesssim \frac{1}{\phi(4P)} \left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset 4P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\}_{j \geq (j_{4P} \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(4P))} \lesssim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}. \end{aligned}$$

To estimate $I_{P,2}$, we only consider the case that $q_+ \in (0, \infty)$, since the proof of the case that $q_+ = \infty$ is similar, the details being omitted. Without loss of generality, we may assume that $\|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} = 1$ and prove that $I_{P,2} \lesssim 1$. To this end, it suffices to show that

$$\left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \frac{\chi_P}{\phi(P)} |Q|^{-\frac{s(\cdot)}{n}} |(u^P)_{r, \lambda}^*|_Q \tilde{\chi}_Q \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} \lesssim 1.$$

By (2.1), and (i) and (ii) of Remark 2.9, we see that the above inequality is equivalent to that there exists some large positive constant C_0 such that

$$\sum_{j=(j_P \vee 0)}^{\infty} \left\| \left[\sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \frac{\chi_P}{C_0 \phi(P)} |Q|^{-\frac{s(\cdot)}{n}} ((u^P)_{r,\lambda}^*)_{Q} \tilde{\chi}_Q \right] \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}^{q(\cdot)} \leq 1,$$

which, by Lemma 3.14(i), is a consequence of

$$(3.6) \quad J_P := \sum_{j=(j_P \vee 0)}^{\infty} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \frac{\chi_P}{C_0 \phi(P)} |Q|^{-\frac{s(\cdot)}{n}} ((u^P)_{r,\lambda}^*)_{Q} \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q-} \leq 1.$$

Now we show (3.6). Since $\|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} = 1$, it follows that, for all $\tilde{P} \in \mathcal{Q}$,

$$\left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset \tilde{P} \\ \ell(Q)=2^{-j}}} [\phi(\tilde{P})]^{-1} \chi_{\tilde{P}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\}_{j \geq (j_{\tilde{P}} \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\tilde{P}))} \leq 1,$$

which, together with (2.1), and (i) and (ii) of Remark 2.9, implies that

$$\sum_{j=(j_{\tilde{P}} \vee 0)}^{\infty} \left\| \left[\sum_{\substack{Q \in \mathcal{Q}^*, Q \subset \tilde{P} \\ \ell(Q)=2^{-j}}} [\phi(\tilde{P})]^{-1} \chi_{\tilde{P}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right] \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}^{q(\cdot)} \leq 1.$$

From this, and (iii) and (iv) of Lemma 3.14, we deduce that, for all $\tilde{P} \in \mathcal{Q}$ and $j \geq (j_{\tilde{P}} \vee 0)$,

$$(3.7) \quad \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset \tilde{P} \\ \ell(Q)=2^{-j}}} [\phi(\tilde{P})]^{-1} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\tilde{P})} \leq 1.$$

For the given $P \in \mathcal{Q}$, $i \in \mathbb{Z}_+$ and $l \in \mathbb{Z}^n$, let

$$A(i, l, P) := \{R \in \mathcal{Q}^* : \ell(R) = 2^{-i} \ell(P), R \subset P + l\ell(P)\}.$$

Then we see that

$$\begin{aligned} \tilde{J}_P &:= \sum_{j=(j_P \vee 0)}^{\infty} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} \chi_P [\phi(P)]^{-1} |Q|^{-\frac{s(\cdot)}{n}} (u_{r,\lambda}^*)_{Q} \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q-} \\ &\lesssim \sum_{i=0}^{\infty} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-i} \ell(P)}} \frac{\chi_P}{\phi(P)} |Q|^{-[\frac{s(\cdot)}{n} + \frac{1}{2}]} \right\| \end{aligned}$$

$$\times \left[\sum_{l \in \mathbb{Z}^n, |l| \geq 2} \sum_{R \in A(i, l, P)} \frac{|u_R|^r}{(1 + \{\ell(Q)\}^{-1} |x_R - x_Q|)^\lambda} \right]^{\frac{1}{r}} \left\| \chi_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-}.$$

Notice that, for all $i \in \mathbb{Z}_+$, $l \in \mathbb{Z}^n$ and $x \in Q \in \mathcal{Q}^*$ with $\ell(Q) = 2^{-i}\ell(P)$,

$$\sum_{R \in A(i, l, P)} \frac{|u_R|^r}{[1 + \{\ell(Q)\}^{-1} |x_R - x_Q|]^m} \sim \eta_{j_Q, m} * \left(\left[\sum_{R \in A(i, l, P)} |u_R| \chi_R \right]^r \right) (x),$$

where $m \in (n + C_{\log}(s), \infty)$ is chosen such that $\lambda > m + n + 2r \log_2 c_1$. Notice that, when $|l| \geq 2$, $1 + \{\ell(Q)\}^{-1} |x_R - x_Q| \sim 2^i |l|$. Thus, by Lemma 3.12, we know that

$$\begin{aligned} \tilde{J}_P &\lesssim \sum_{i=0}^{\infty} \left\| \sum_{\substack{l \in \mathbb{Z}^n \\ |l| \geq 2}} \frac{(2^i |l|)^{m-\lambda}}{[\phi(P)]^r} \eta_{i+j_P, m-C_{\log}(s)} * \left(\left[\sum_{R \in A(i, l, P)} |R|^{-\frac{s(\cdot)}{n} r} |u_R| \tilde{\chi}_R \right]^r \right) \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}^{\frac{q_-}{r}} \\ &\lesssim \sum_{i=0}^{\infty} \left\{ \sum_{\substack{l \in \mathbb{Z}^n \\ |l| \geq 2}} (2^i |l|)^{m-\lambda} \left[\frac{\phi(P + l\ell(P))}{\phi(P)} \right]^r \left\| \sum_{R \in A(i, l, P)} \frac{\chi_{P+l\ell(P)}}{\phi(P + l\ell(P))} |R|^{-\frac{s(\cdot)}{n}} |t_R| \tilde{\chi}_R \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r \right\}^{\frac{q_-}{r}}, \end{aligned}$$

which, combined with (3.7) and Lemma 3.6, implies that

$$\tilde{J}_P \lesssim \sum_{i=0}^{\infty} \left\{ \sum_{l \in \mathbb{Z}^n, |l| \geq 2} 2^{i(m-\lambda)} |l|^{m+2r \log_2 c_1 - \lambda} \right\}^{\frac{q_-}{r}} \sim 1.$$

Therefore, there exists a positive constant C_0 large enough such that (3.6) holds true for all $P \in \mathcal{Q}$ and hence

$$(3.8) \quad \mathbf{I}_{P,2} \lesssim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

Combining (3.4), (3.5) and (3.8), we conclude that

$$\|t_{r, \lambda}^*\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \leq \sup_{P \in \mathcal{Q}} (\mathbf{I}_{P,1} + \mathbf{I}_{P,2}) \lesssim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)},$$

which completes the proof of Lemma 3.15. \square

Now we come to prove the main result of this section.

Proof of Theorem 3.3. We first show that S_φ is bounded from $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ to $b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$. Let $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, $r \in (0, \frac{1}{2} \min\{p_-, q_-, 2\})$ and $m \in (n + C_{\log}(s) + C_{\log}(r/q) + \log_2 c_1, \infty)$. Then, by Lemma 3.9, we see that, for all $Q_{jk} \in \mathcal{Q}^*$ and $x \in Q_{jk}$,

$$|\varphi_j * f(x_{Q_{jk}})|^r \lesssim 2^{jn} \sum_{l \in \mathbb{Z}^n} \int_{Q_{j(k+l)}} \frac{|\varphi_j * f(y)|^r}{(1 + 2^j |x - y|)^{4m}} dy,$$

which, together with the fact that $1 + 2^j |x - y| \sim 1 + |l|$ when $x \in Q_{jk}$ and $y \in Q_{j(k+l)}$, implies that

$$|\varphi_j * f(x_{Q_{jk}})| \lesssim \left[\sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-m} \eta_{j, 3m} * |(\varphi_j * f) \chi_{Q_{j(k+l)}}|^r(x) \right]^{\frac{1}{r}}.$$

From this, Lemma 3.10 and Remark 2.9(iv), we deduce that

$$\begin{aligned}
& \|S_\varphi(f)\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \\
& \lesssim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{k \in \mathbb{Z}^n} \left[\sum_{l \in \mathbb{Z}^n} \frac{2^{jrs(\cdot)}}{(1+|l|)^m} \eta_{j,3m} * |(\varphi_j * f)\chi_{3n|l|P}|^r \right]^{\frac{1}{r}} \chi_{Q_{jk}} \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
& \lesssim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{l \in \mathbb{Z}^n} \frac{2^{jrs(\cdot)}}{(1+|l|)^m} \eta_{j,3m} * |(\varphi_j * f)\chi_{3n|l|P}|^r \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}}(P))}^{\frac{1}{r}} \\
& \lesssim \left[\sum_{l \in \mathbb{Z}^n} (1+|l|)^{-m} \sup_{P \in \mathcal{Q}} \frac{1}{\{\phi(P)\}^r} \left\| \left\{ \eta_{j,2m} * |2^{js(\cdot)}(\varphi_j * f)\chi_{3n|l|P}|^r \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}}(P))} \right]^{\frac{1}{r}},
\end{aligned}$$

which, combined with Lemmas 3.12 and 3.6, implies that

$$\begin{aligned}
\|S_\varphi(f)\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} & \lesssim \left[\sum_{l \in \mathbb{Z}^n} (1+|l|)^{-m} \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)^r} \left\| \left\{ 2^{js(\cdot)}|\varphi_j * f| \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(3n|l|P))}^r \right]^{\frac{1}{r}} \\
& \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{l \in \mathbb{Z}^n} (1+|l|)^{-m} (1+|l|)^{r \log_2 c_1} \right\}^{\frac{1}{r}} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, S_φ is bounded from $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ to $b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$.

The boundedness of T_ψ from $b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ to $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ is deduced from an argument similar to that used in the proof of [78, Theorem 2.1]. Indeed, by repeating the argument used in the proof of [78, Theorem 2.1], with [78, Lemmas 2.7 and 2.8] therein replaced by Lemmas 3.8 and 3.15, we conclude that T_ψ is bounded from $b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ to $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, the details being omitted. Finally, by the Calderón reproducing formula (see, for example, [78, Lemma 2.3]), we know that $T_\psi \circ S_\varphi$ is the identity on $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, which completes the proof of Theorem 3.3. \square

4 Embeddings

In this section, we prove some basic properties and embeddings between $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Recall that the *Triebel-Lizorkin-type space with variable exponents*, $F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{j=\max\{j_P, 0\}}^{\infty} \left[2^{js(\cdot)}|\varphi_j * f(\cdot)| \right]^{q(\cdot)} \right\}^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}(P)} < \infty,$$

where φ_0 is replaced by Φ and the supremum is taken over all dyadic cubes P in \mathbb{R}^n , which was introduced in [76].

Proposition 4.1. *Let $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$, $s, s_0, s_1 \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $p, q, q_0, q_1 \in C^{\log}(\mathbb{R}^n)$.*

- (i) *If $q_0 \leq q_1$, then $B_{p(\cdot),q_0(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot),q_1(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$.*

- (ii) If $(s_0 - s_1)_- > 0$, then $B_{p(\cdot), q_0(\cdot)}^{s_0(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), q_1(\cdot)}^{s_1(\cdot), \phi}(\mathbb{R}^n)$.
 (iii) If $p_+, q_+ \in (0, \infty)$, then

$$B_{p(\cdot), \min\{p(\cdot), q(\cdot)\}}^{s(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow F_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), \max\{p(\cdot), q(\cdot)\}}^{s(\cdot), \phi}(\mathbb{R}^n).$$

In particular, if $p_+ \in (0, \infty)$, then $B_{p(\cdot), p(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) = F_{p(\cdot), p(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$.

Proof. The proof of this proposition is similar to that of [3, Theorem 6.1] and we only give the proof of (iii). Let $f_j(x) := 2^{js(x)}|\varphi_j * f(x)|$ for all $x \in \mathbb{R}^n$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{Z}_+$. To prove the first embedding of (iii), we let $r(\cdot) := \min\{p(\cdot), q(\cdot)\}$ and $f \in B_{p(\cdot), r(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|f\|_{B_{p(\cdot), r(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} = 1$ and prove that $\|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim 1$. Obviously, for all $P \in \mathcal{Q}$,

$$\left\| \{[\phi(P)]^{-1} \chi_P f_j\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{r(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} \leq 1,$$

which, together with (2.1), Remarks 2.7(i) and 2.9(i), implies that

$$\left\| \sum_{j=(j_P \vee 0)}^{\infty} [\phi(P)]^{-1} \chi_P f_j \right\|_{L^{\frac{p(\cdot)}{r(\cdot)}}(\mathbb{R}^n)}^{r(\cdot)} \leq \sum_{j=(j_P \vee 0)}^{\infty} \left\| \{[\phi(P)]^{-1} \chi_P f_j\}^{r(\cdot)} \right\|_{L^{\frac{p(\cdot)}{r(\cdot)}}(\mathbb{R}^n)} \leq 1.$$

Then, by Remark 2.5 and the fact that, for all $d \in (0, 1]$ and $\{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$,

$$(4.1) \quad \left(\sum_{j \in \mathbb{N}} |a_j| \right)^d \leq \sum_{j \in \mathbb{N}} |a_j|^d,$$

we find that, for all $P \in \mathcal{Q}$,

$$\varrho_{p(\cdot)} \left(\left[\sum_{j=(j_P \vee 0)}^{\infty} \{[\phi(P)]^{-1} \chi_P f_j\}^{q(\cdot)} \right]^{\frac{1}{q(\cdot)}} \right) \leq \varrho_{\frac{p(\cdot)}{r(\cdot)}} \left(\sum_{j=(j_P \vee 0)}^{\infty} \{[\phi(P)]^{-1} \chi_P f_j\}^{r(\cdot)} \right) \leq 1,$$

which implies that

$$\frac{1}{\phi(P)} \left\| \left\{ \sum_{j=(j_P \vee 0)}^{\infty} [2^{js(\cdot)} |\varphi_j * f|]^{q(\cdot)} \right\}^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}(P)} \leq 1.$$

Therefore, $\|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \leq 1$, which completes the proof of the first embedding of (iii).

For the second embedding of (iii), let $f \in F_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ and $\alpha(\cdot) := \max\{p(\cdot), q(\cdot)\}$. Without loss of generality, we may assume that $\|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} = 1$ and show that $\|f\|_{B_{p(\cdot), \alpha(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim 1$. Since $\|f\|_{F_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} = 1$, we know that, for all $P \in \mathcal{Q}$,

$$\left\| \left\{ \sum_{j=(j_P \vee 0)}^{\infty} ([\phi(P)]^{-1} \chi_P f_j)^{q(\cdot)} \right\}^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1,$$

which, combined with (4.1) and Remark 2.5, implies that, for all $P \in \mathcal{Q}$,

$$\varrho_{\frac{p(\cdot)}{\alpha(\cdot)}} \left(\sum_{j=(j_P \vee 0)}^{\infty} \{[\phi(P)]^{-1} \chi_P f_j\}^{\alpha(\cdot)} \right) \leq \varrho_{p(\cdot)} \left(\left[\sum_{j=(j_P \vee 0)}^{\infty} \{[\phi(P)]^{-1} \chi_P f_j\}^{q(\cdot)} \right]^{\frac{1}{q(\cdot)}} \right) \leq 1.$$

From this, Remark 2.9(ii) and Remark 2.7(iv), we deduce that

$$\begin{aligned} \varrho_{\ell^{\alpha(\cdot)}(L^{p(\cdot)})} \left(\{[\phi(P)]^{-1} \chi_P f_j\}_{j \geq (j_P \vee 0)} \right) &= \sum_{j=(j_P \vee 0)}^{\infty} \left\| ([\phi(P)]^{-1} \chi_P f_j)^{\alpha(\cdot)} \right\|_{L^{\frac{p(\cdot)}{\alpha(\cdot)}}(\mathbb{R}^n)} \\ &\leq \left\| \sum_{j=(j_P \vee 0)}^{\infty} ([\phi(P)]^{-1} \chi_P f_j)^{\alpha(\cdot)} \right\|_{L^{\frac{p(\cdot)}{\alpha(\cdot)}}(\mathbb{R}^n)} \leq 1, \end{aligned}$$

which implies that $\|f\|_{B_{p(\cdot), \alpha(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \leq 1$ and hence completes the proof of Lemma 4.1. \square

The Sobolev-type embedding of $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ (see [3, Theorem 6.4]) shows that it is reasonable and necessary to consider the Besov spaces with variable smoothness and integrability. For $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, we also have the following Sobolev-type embeddings.

Proposition 4.2. *Let $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$, $s_0, s_1 \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $p_0, p_1 \in C^{\log}(\mathbb{R}^n)$ satisfy that, for all $x \in \mathbb{R}^n$, $s_1(x) \leq s_0(x)$ and $s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$. Then*

$$(4.2) \quad b_{p_0(\cdot), \infty}^{s_0(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow b_{p_1(\cdot), \infty}^{s_1(\cdot), \phi}(\mathbb{R}^n);$$

moreover, $B_{p_0(\cdot), \infty}^{s_0(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow B_{p_1(\cdot), \infty}^{s_1(\cdot), \phi}(\mathbb{R}^n)$.

Proof. To prove this proposition, we only need to show (4.2), since the embedding $B_{p_0(\cdot), \infty}^{s_0(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow B_{p_1(\cdot), \infty}^{s_1(\cdot), \phi}(\mathbb{R}^n)$ is a consequence of (4.2) and Theorem 3.3. To prove (4.2), let $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \in b_{p_0(\cdot), \infty}^{s_0(\cdot), \phi}(\mathbb{R}^n)$ and $P \in \mathcal{Q}$ be any given dyadic cube. For all $Q \in \mathcal{Q}^*$, let $u_Q := t_Q$ when $Q \subset P$ and $u_Q = 0$ otherwise. Then, by the Sobolev-type embedding of $b_{p(\cdot), \infty}^{s(\cdot)}(\mathbb{R}^n) = b_{p(\cdot), \infty}^{s(\cdot), 1}(\mathbb{R}^n)$ ([28, Proposition 3.9]), namely, $b_{p_0(\cdot), \infty}^{s_0(\cdot), 1}(\mathbb{R}^n) \hookrightarrow b_{p_1(\cdot), \infty}^{s_1(\cdot), 1}(\mathbb{R}^n)$, we conclude that

$$\begin{aligned} &\sup_{j \geq (j_P \vee 0)} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s_1(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p_1(\cdot)}(P)} \\ &= \sup_{j \geq 0} \left\| \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-j}} |Q|^{-\frac{s_1(\cdot)}{n}} |u_Q| \tilde{\chi}_Q \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} = \|u\|_{b_{p_1(\cdot), \infty}^{s_1(\cdot), 1}(\mathbb{R}^n)} \\ &\lesssim \|u\|_{b_{p_0(\cdot), \infty}^{s_0(\cdot), 1}(\mathbb{R}^n)} \sim \sup_{j \geq 0} \left\| \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-j}} |Q|^{-\frac{s_0(\cdot)}{n}} |u_Q| \tilde{\chi}_Q \right\|_{L^{p_0(\cdot)}(\mathbb{R}^n)} \\ &\sim \sup_{j \geq (j_P \vee 0)} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s_0(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p_0(\cdot)}(P)}. \end{aligned}$$

From this, we further deduce that

$$\begin{aligned} \|t\|_{b_{p_1(\cdot),\infty}^{s_1(\cdot),\phi}(\mathbb{R}^n)} &= \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \geq (j_P \vee 0)} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s_1(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p_1(\cdot)}(P)} \\ &\lesssim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \geq (j_P \vee 0)} \left\| \sum_{\substack{Q \in \mathcal{Q}^*, Q \subset P \\ \ell(Q)=2^{-j}}} |Q|^{-\frac{s_0(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p_0(\cdot)}(P)} \sim \|t\|_{b_{p_0(\cdot),\infty}^{s_0(\cdot),\phi}(\mathbb{R}^n)}, \end{aligned}$$

which implies that (4.2) holds true and hence completes the proof of Proposition 4.2. \square

Theorem 4.3. *Let $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$, $s_0, s_1 \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $p_0, p_1, q \in C^{\log}(\mathbb{R}^n)$. Assume that, for all $x \in \mathbb{R}^n$, $s_1(x) \leq s_0(x)$ and*

$$(4.3) \quad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}.$$

Then $B_{p_0(\cdot),q(\cdot)}^{s_0(\cdot),\phi}(\mathbb{R}^n) \hookrightarrow B_{p_1(\cdot),q(\cdot)}^{s_1(\cdot),\phi}(\mathbb{R}^n)$.

Proof. We only give the proof of the case that $q_+ \in (0, \infty)$, since the case that $q_+ = \infty$ was proved in Proposition 4.2. Let $f \in B_{p_0(\cdot),q(\cdot)}^{s_0(\cdot),\phi}(\mathbb{R}^n)$ and, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $g_j(x) := \varphi_j * f(x)$. Without loss of generality, we may assume that $\|f\|_{B_{p_0(\cdot),q(\cdot)}^{s_0(\cdot),\phi}(\mathbb{R}^n)} = 1$. Next, we show that $\|f\|_{B_{p_1(\cdot),q(\cdot)}^{s_1(\cdot),\phi}(\mathbb{R}^n)} \lesssim 1$. Obviously, by Remark 3.2, (2.1), and (i) and (ii) of Remark 2.9, we find that, for all $R \in \mathcal{D}_0(\mathbb{R}^n)$,

$$(4.4) \quad \sum_{j=(j_R \vee 0)}^{\infty} \left\| \left[\frac{\chi_R}{\phi(R)} 2^{js_0(\cdot)} |g_j| \right]^{q(\cdot)} \right\|_{L^{\frac{p_0(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \lesssim 1.$$

Let $P \in \mathcal{Q}$ be a given dyadic cube. We claim that there exists $c \in (0, 1)$, independent of P , such that, for all $j \geq [j_P \vee 0, \infty)$,

$$\left\| \left[\frac{c\chi_P}{\phi(P)} 2^{js_1(\cdot)} |g_j| \right]^{q(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \leq \sum_{i=1}^{\infty} 2^{-i\xi} \left\| \left[\frac{\chi_{P_i}}{\phi(P_i)} 2^{js_0(\cdot)} |g_j| \right]^{q(\cdot)} \right\|_{L^{\frac{p_0(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} + 2^{-j} =: \delta_j,$$

where $P_i := 2^{i+1+n}P$ and $\xi \in (0, \infty)$. From this claim and (4.4), we deduce that

$$\sum_{j=(j_P \vee 0)}^{\infty} \left\| \left[\frac{c\chi_P}{\phi(P)} 2^{js_1(\cdot)} |g_j| \right]^{q(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \lesssim 1,$$

which, together with (2.1), and (i) and (ii) of Remark 2.9, implies that

$$\|f\|_{B_{p_1(\cdot),q(\cdot)}^{s_1(\cdot),\phi}(\mathbb{R}^n)} \lesssim 1 \sim \|f\|_{B_{p_0(\cdot),q(\cdot)}^{s_0(\cdot),\phi}(\mathbb{R}^n)}.$$

Therefore, it remains to prove the above claim. Observe that, for all $j \geq [j_P \vee 0, \infty)$, $\delta_j \in [2^{-j}, 2^{-j} + \theta]$ with $\theta \in [0, \infty)$. Then, by Lemma 3.9 and Remark 3.11, we conclude that, for all $x \in \mathbb{R}^n$, $r \in (0, p_-)$ and $m \in (0, \infty)$ large enough,

$$(4.5) \quad \frac{2^{jr[s_1(x) - \frac{n}{p_1(x)}]}}{[\phi(P)]^r \delta_j^{r/q(x)}} |g_j(x)|^r \lesssim \eta_{j,2m} * \left(\left\{ \frac{2^{j[s_1(\cdot) - \frac{n}{p_1(\cdot)}]}}{\phi(P) \delta_j^{1/q(\cdot)}} |g_j| \right\}^r \right) (x)$$

$$\lesssim \sum_{i=1}^{\infty} \int_{D_{i,P}} \frac{2^{jn} (2^{j[s_1(y) - \frac{n}{p_1(y)}]} |g_j(y)|)^r}{[\phi(P)]^r \delta_j^{\frac{r}{q(y)}} (1 + 2^j |x - y|)^{2m}} dy =: \sum_{i=1}^{\infty} A_{j,i}(x),$$

where $D_{1,P} := 4\sqrt{n}P$ and, for all $i \in [2, \infty)$, $D_{i,P} := (2^{i+1}\sqrt{n}P) \setminus (2^i\sqrt{n}P)$. For $A_{j,1}$, by the Hölder inequality in Remark 2.7(ii), (4.3), (2.4) and Lemma 3.14, we see that

$$\begin{aligned} A_{j,1} &\lesssim \left\| \left[\frac{\chi_{4\sqrt{n}P}}{\phi(P)\delta_j^{1/q(\cdot)}} 2^{js_0(\cdot)} |g_j| \right]^r \right\|_{L^{\frac{p_0(\cdot)}{r}}(\mathbb{R}^n)} \left\| \frac{2^{jn} 2^{-jnr/p_0(\cdot)}}{(1 + 2^j |x - \cdot|)^{2m}} \right\|_{L^{(\frac{p_0(\cdot)}{r})^*}(\mathbb{R}^n)} \\ &\lesssim \left[\frac{\phi(P_1)}{\phi(P)} \right]^r \left\| \frac{\chi_{4\sqrt{n}P}}{\phi(P_1)\delta_j^{1/q(\cdot)}} 2^{js_0(\cdot)} |g_j| \right\|_{L^{p_0(\cdot)}(\mathbb{R}^n)}^r \lesssim 1, \end{aligned}$$

where the last inequality follows from the definition of δ_j . Similarly, observe that, for all $x \in P$ and $y \in D_{i,P}$ with $i \geq 2$, $|x - y| \gtrsim 2^{i-j_P}$, then the fact that $j \geq j_P$ further implies that

$$\begin{aligned} A_{j,i} &\lesssim \frac{2^{(j_P-j)m}}{2^{i(m-\xi/q_-)}} \left[\frac{\phi(P_i)}{\phi(P)} \right]^r \left\| \frac{\chi_{P_i} 2^{js_0(\cdot)} |g_j|}{\phi(P_i)\{2^{\xi}\delta_j\}^{1/q(\cdot)}} \right\|_{L^{p_0(\cdot)}(\mathbb{R}^n)}^r \left\| \frac{2^{jn} 2^{-jnr/p_0(\cdot)}}{(1 + 2^j |x - \cdot|)^m} \right\|_{L^{(\frac{p_0(\cdot)}{r})^*}(\mathbb{R}^n)} \\ &\lesssim 2^{(j_P-j)m} 2^{-i(m-r \log_2 c_1)} \lesssim 2^{-i(m-\xi/q_- - r \log_2 c_1)}. \end{aligned}$$

Thus, by (4.5), we conclude that, for all $x \in \mathbb{R}^n$,

$$\chi_P(x) [\phi(P)]^{-1} \delta_j^{-1/q(x)} 2^{j[s_1(x) - \frac{n}{p_1(x)}]} |g_j(x)| \lesssim 1.$$

From this, (4.3) and an appropriate choice of $c \in (0, 1)$, we deduce that

$$\begin{aligned} &\left[\frac{c\chi_P(x) 2^{js_1(x)}}{\phi(P)\delta_j^{1/q(x)}} |g_j(x)| \right]^{p_1(x)} \\ &= c^{p_0(x)} \left[\frac{\chi_P(x) 2^{js_0(x)}}{\phi(P)\delta_j^{1/q(x)}} |g_j(x)| \right]^{p_0(x)} \left[\frac{c\chi_P(x) 2^{j[s_1(x) - \frac{n}{p_1(x)}]}}{\phi(P)\delta_j^{1/q(x)}} |g_j(x)| \right]^{p_1(x) - p_0(x)} \\ &\leq c^{p_0(x)} \left[\frac{\chi_P(x) 2^{js_0(x)}}{\phi(P)\delta_j^{1/q(x)}} |g_j(x)| \right]^{p_0(x)} \leq \left[\frac{\chi_{P_1}(x) 2^{js_0(x)}}{\phi(P_1)\{2^{\xi}\delta_j\}^{1/q(x)}} |g_j(x)| \right]^{p_0(x)}, \end{aligned}$$

which, together with the definition of δ_j and Remark 2.9(ii), implies that the previous claim holds true and hence completes the proof of Theorem 4.3. \square

Remark 4.4. When $\phi \equiv 1$, Theorem 4.3 just becomes [3, Theorem 6.4], which is called the Sobolev inequality therein.

5 Equivalent quasi-norms

In this section, we are aimed to characterize $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ in terms of the Peetre maximal functions and establish their atomic characterization via Sobolev embeddings. Following [17, p. 19], for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $a \in (0, \infty)$ and $s : \mathbb{R}^n \rightarrow \mathbb{R}$, the *Peetre maximal function* of f is defined by setting, for all $j \in \mathbb{Z}_+$,

$$\varphi_j^{*,a}(2^{js(\cdot)} f)(x) := \sup_{y \in \mathbb{R}^n} \frac{2^{js(y)} |\varphi_j * f(y)|}{(1 + 2^j |x - y|)^a}.$$

The following Theorem 5.1 is the first main result of this section.

Theorem 5.1. *Let p, q, s, ϕ be as in Definition 2.12 and*

$$(5.1) \quad a \in ([n + \log_2 c_1]/p_-, \infty).$$

Then $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}^ < \infty$, where*

$$\|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}^* := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \varphi_j^{*, a}(2^{js(\cdot)} f) \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}.$$

Moreover, for all $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, $\|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \sim \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}^$ with equivalent positive constants independent of f .*

Remark 5.2. Theorem 5.1 goes back to [17, Theorem 1] when $\phi \equiv 1$.

To prove Theorem 5.1, we need some technical lemmas. For all $r \in (0, \infty)$, denote by $L_{\text{loc}}^r(\mathbb{R}^n)$ the set of all r -locally integrable functions on \mathbb{R}^n . Recall that the Hardy-Littlewood maximal operator \mathcal{M} is defined by setting, for all $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B of \mathbb{R}^n containing x .

The following Lemma 5.3 is just [15, Theorem 4.3.8].

Lemma 5.3. *Let $p \in C^{\log}(\mathbb{R}^n)$ with $p_- \in (1, \infty]$. Then there exists a positive constant C , independent of f , such that, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathcal{M}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.*

The following technical lemma plays a key role in the proof of Theorem 5.1.

Lemma 5.4. *Let p, q, s, ϕ be as in Definition 2.12 and $a \in (n + \log_2 c_1 + \varepsilon/q_-, \infty)$ with $\varepsilon \in (0, \infty)$. Assume that $p_- \in (1, \infty)$, $q_+ \in (0, \infty)$ and $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ with norm 1. Then there exists a positive constant c such that, for all $P \in \mathcal{Q}$ and $j \in \mathbb{Z}_+$ with $j \geq (j_P \vee 0)$,*

$$(5.2) \quad \inf \left\{ \lambda_j \in (0, \infty) : \varrho_{p(\cdot)} \left(\frac{c \chi_P \varphi_j^{*, a}(2^{js(\cdot)} f)}{\phi(P) \lambda_j^{1/q(\cdot)}} \right) \leq 1 \right\} \\ \leq \sum_{k=1}^{\infty} \inf \left\{ \eta_j \in (0, \infty) : \varrho_{p(\cdot)} \left(\frac{\chi_{P_k^n} 2^{js(\cdot)} |\varphi_j * f|}{2^{k\varepsilon/q(\cdot)} \phi(P_k^n) \eta_j^{1/q(\cdot)}} \right) \leq 1 \right\} + 2^{-\sigma[j - (j_P \vee 0)]},$$

where, for all $k \in \mathbb{N}$, $P_k^n := 2^{k+1+n}P$ and $\sigma \in (0, \frac{a-n}{4(1/q_- - 1/q_+)})$.

Proof. Let δ_j^P be the right hand side term of (5.2). Then, by Remark 3.2, we easily see that

$$\delta_j^P \leq \sum_{k=1}^{\infty} 2^{-k\varepsilon} \frac{1}{\phi(P_k^n)} \left\| \left\{ 2^{js(\cdot)} |\varphi_j * f| \right\}_{j \geq (j_{P_k^n} \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P_k^n))} + 2^{-\sigma[j - (j_P \vee 0)]} \\ \leq \sum_{k=1}^{\infty} 2^{-k\varepsilon} \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} + 2^{-\sigma[j - (j_P \vee 0)]} = 1/(2^\varepsilon - 1) + 2^{-\sigma[j - (j_P \vee 0)]},$$

which implies that

$$(5.3) \quad \delta_j^P \in \left[2^{-\sigma[j - (j_P \vee 0)]}, 1/(2^\varepsilon - 1) + 2^{-\sigma[j - (j_P \vee 0)]} \right].$$

Thus, to prove Lemma 5.4, we only need to show that, for some positive constant c ,

$$\inf \left\{ \lambda_j \in (0, \infty) : \varrho_{p(\cdot)} \left(\frac{c \chi_P (\delta_j^P)^{-1/q(\cdot)} \varphi_j^{*,a} (2^{js(\cdot)} f)}{\phi(P) \lambda_j^{1/q(\cdot)}} \right) \leq 1 \right\} \leq 1,$$

which, via Remark 2.9(ii), and (i) and (ii) of Lemma 3.14, is a consequence of

$$(5.4) \quad H_P := \left\| \frac{\chi_P (\delta_j^P)^{-1/q(\cdot)}}{\phi(P)} \varphi_j^{*,a} (2^{js(\cdot)} f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1.$$

Next we prove (5.4). By Lemma 3.9 and the inequality that, for all $x, y, z \in \mathbb{R}^n$,

$$(1 + 2^{-j}|x - y|)^{-a} \leq (1 + 2^{-j}|x - z|)^{-a} (1 + 2^{-j}|z - y|)^a,$$

we find that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} \varphi_j^{*,a} (2^{js(\cdot)} f)(x) &\lesssim \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{2^{jn} 2^{js(z)} |\varphi_j * f(z)|}{(1 + 2^j|y - z|)^{2a}} dz \frac{1}{(1 + 2^j|x - y|)^a} \\ &\lesssim \int_{\mathbb{R}^n} \frac{2^{jn} 2^{js(z)} |\varphi_j * f(z)|}{(1 + 2^j|x - z|)^a} dz \\ &\sim \int_{4\sqrt{n}P} \frac{2^{jn} 2^{js(z)} |\varphi_j * f(z)|}{(1 + 2^j|x - z|)^a} dz + \sum_{k=2}^{\infty} \int_{D_{k,P}} \cdots dz =: A_j(x) + \sum_{k=2}^{\infty} A_j^k(x), \end{aligned}$$

where, for all $k \in \mathbb{N} \cap [2, \infty)$, $D_{k,P} := (2^{k+1}\sqrt{n}P) \setminus (2^k\sqrt{n}P)$. Thus, we obtain

$$(5.5) \quad H_P \leq \left\| \frac{\chi_P A_j(\cdot)}{[\delta_j^P]^{1/q(\cdot)} \phi(P)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \frac{\chi_P}{[\delta_j^P]^{1/q(\cdot)} \phi(P)} \sum_{k=2}^{\infty} A_j^k(\cdot) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} =: H_{P,1} + H_{P,2}.$$

We first estimate $H_{P,1}$. For all $x \in P$, we write

$$(5.6) \quad A_j(x) \sim \left\{ \int_{B_{-1}^j(x)} + \sum_{i=0}^{\infty} \int_{B_i^j(x)} \right\} \frac{2^{jn} 2^{js(z)} |\varphi_j * f(z)| \chi_{4\sqrt{n}P}(z)}{(1 + 2^j|x - z|)^a} dz =: A_{j,1}(x) + A_{j,2}(x),$$

where, for all $x \in \mathbb{R}^n$, $B_{-1}^j(x) := B(x, 2^{-[j-(j_P \vee 0)]/2})$ and, for all $i \in \mathbb{Z}_+$,

$$B_i^j(x) := B\left(x, 2^{-[j-(j_P \vee 0)]/2+i+1}\right) \setminus B\left(x, 2^{-[j-(j_P \vee 0)]/2+i}\right).$$

From (5.3), $q \in C^{\log}(\mathbb{R}^n)$ and Remark 2.10(ii), we deduce that, for all $x \in \mathbb{R}^n$ and $z \in B_{-1}^j(x)$,

$$\begin{aligned} (\delta_j^P)^{\frac{1}{q(z)} - \frac{1}{q(x)}} &\leq \left\{ 2^{\sigma[j-(j_P \vee 0)]} \delta_j^P \right\}^{\left| \frac{1}{q(z)} - \frac{1}{q(x)} \right|} \left\{ 2^{\sigma[j-(j_P \vee 0)]} \right\}^{\left| \frac{1}{q(z)} - \frac{1}{q(x)} \right|} \\ &\lesssim 2^{2\sigma[j-(j_P \vee 0)]C_{\log}(1/q)/\log(e+1/|x-z|)} \lesssim 1. \end{aligned}$$

By this, $a \in (n, \infty)$ and [59, p. 59, (3.9)], we conclude that, for all $x \in P$,

$$(5.7) \quad \frac{(\delta_j^P)^{-1/q(x)}}{\phi(P)} A_{j,1}(x) \lesssim \frac{1}{\phi(P)} \int_{B_{-1}^j(x)} \frac{2^{jn} 2^{js(z)} |\varphi_j * f(z)| \chi_{4\sqrt{n}P}(z)}{[\delta_j^P]^{1/q(z)} (1 + 2^j|x - z|)^a} dz$$

$$\lesssim \mathcal{M} \left(\frac{2^{js(\cdot)} |\varphi_j * f| \chi_{4\sqrt{n}P}}{[\delta_j^P]^{1/q(\cdot)} \phi(P)} \right) (x).$$

On the other hand, by (5.3), we see that, for all $x \in P$ and $z \in \mathbb{R}^n$ with $i \in \mathbb{Z}_+$,

$$(5.8) \quad (\delta_j^P)^{[\frac{1}{q(z)} - \frac{1}{q(x)}]} \lesssim 2^{2\sigma[j - (j_P \vee 0)](\frac{1}{q_-} - \frac{1}{q_+})}$$

and $1 + 2^j |x - z| \geq 1 + 2^j 2^{-\frac{j - (j_P \vee 0)}{2} + i}$. Thus, by $\sigma \in (0, \frac{a-n}{4(1/q_- - 1/q_+)})$, we conclude that, for all $x \in P$,

$$\begin{aligned} \frac{(\delta_j^P)^{-\frac{1}{q(x)}}}{\phi(P)} A_{j,2}(x) &\lesssim \sum_{i=0}^{\infty} \frac{2^{2\sigma[j - (j_P \vee 0)](\frac{1}{q_-} - \frac{1}{q_+})}}{\phi(P) 2^{[\frac{j + (j_P \vee 0)}{2} + i]a}} \int_{B_i^j(x)} \frac{2^{jn} 2^{js(z)}}{(\delta_j^P)^{1/q(z)}} |\varphi_j * f(z)| \chi_{4\sqrt{n}P}(z) dz \\ &\lesssim 2^{j[2\sigma(\frac{1}{q_-} - \frac{1}{q_+}) + \frac{n}{2} - \frac{a}{2}]} 2^{(j_P \vee 0)[-2\sigma(\frac{1}{q_-} - \frac{1}{q_+}) - \frac{a}{2} + \frac{n}{2}]} \\ &\quad \times \sum_{i=0}^{\infty} 2^{i(n-a)} \mathcal{M} \left(\frac{\chi_{4\sqrt{n}P}}{[\delta_j^P]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f| \right) (x) \\ &\lesssim \mathcal{M} \left(\frac{\chi_{4\sqrt{n}P}}{[\delta_j^P]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f| \right) (x), \end{aligned}$$

which, together with (5.6) and (5.7), implies that, for all $x \in P$,

$$\frac{(\delta_j^P)^{-1/q(x)}}{\phi(P)} A_j(x) \lesssim \mathcal{M} \left(\frac{\chi_{4\sqrt{n}P}}{[\delta_j^P]^{1/q(\cdot)} \phi(P)} 2^{js(\cdot)} |\varphi_j * f| \right) (x).$$

By this, Lemma 5.3 and (2.4), we further know that

$$(5.9) \quad \begin{aligned} H_{P,1} &\lesssim \left\| \frac{\chi_{2^{n+2}P}}{[\delta_j^P]^{1/q(\cdot)} \phi(2^{n+2}P)} 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim 2^{\varepsilon/q_-} \left\| \frac{\chi_{2^{n+2}P} 2^{-\varepsilon/q(\cdot)}}{[\delta_j^P]^{1/q(\cdot)} \phi(2^{n+2}P)} 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1, \end{aligned}$$

where the last inequality comes from the definition of δ_j^P .

We now estimate $H_{P,2}$. Notice that, when $x \in P$ and $z \in D_{k,P}$ with $k \in \mathbb{N} \cap [2, \infty)$, $1 + 2^j |x - z| \gtrsim 2^k 2^{j-j_P}$. Then, by (5.8) and (5.2), we see that, for all $x \in P$,

$$\begin{aligned} (\delta_j^P)^{-1/q(x)} A_j^k(x) &\lesssim 2^{2\sigma[j - (j_P \vee 0)](\frac{1}{q_-} - \frac{1}{q_+})} 2^{-(k+j-j_P)a} 2^{jn} 2^{k\varepsilon/q_-} \int_{D_{k,P}} \frac{2^{-k\varepsilon/q(z)}}{[\delta_j^P]^{1/q(z)}} 2^{js(z)} |\varphi_j * f(z)| dz \\ &\lesssim 2^{-(j-j_P)[a-n-2\sigma(\frac{1}{q_-} - \frac{1}{q_+})]} 2^{-k(a-n-\frac{\varepsilon}{q_-})} \mathcal{M} \left(\frac{\chi_{P_k^n} 2^{-k\varepsilon/q(\cdot)}}{[\delta_j^P]^{1/q(\cdot)}} 2^{js(\cdot)} |\varphi_j * f| \right) (x) \\ &\lesssim 2^{-k(a-n-\frac{\varepsilon}{q_-})} \mathcal{M} \left(\frac{\chi_{P_k^n} 2^{-k\varepsilon/q(\cdot)}}{[\delta_j^P]^{1/q(\cdot)}} 2^{js(\cdot)} |\varphi_j * f| \right) (x), \end{aligned}$$

which, combined with Lemma 5.3, (2.4), the definition of δ_j^P and $a \in (n + \log_2 c_1 + \varepsilon/q_-, \infty)$, implies that

$$(5.10) \quad H_{P,2} \lesssim \sum_{k=2}^{\infty} 2^{-k(a-n-\frac{\varepsilon}{q_-}-\log_2 c_1)} \left\| \frac{\chi_{P_k^n} 2^{-k\varepsilon/q(\cdot)}}{\phi(P_k^n) [\delta_j^P]^{1/q(\cdot)}} 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1.$$

Combining (5.5), (5.9) and (5.10), we conclude that (5.4) holds true and then complete the proof of Lemma 5.4. \square

Proof of Theorem 5.1. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* < \infty$. Then, by the obvious fact that, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$,

$$2^{js(x)}|\varphi_j * f(x)| \leq \varphi_j^{*,a}(2^{js(\cdot)}f)(x),$$

we find that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^*$ and hence $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Thus, to complete the proof of this theorem, we only need to show that, for all $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$,

$$(5.11) \quad \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

Without loss of generality, to prove (5.11), we may assume that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} = 1$ and show that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^* \lesssim 1$. By (5.1), we find that there exist $t \in (0, p_-)$ and $\varepsilon \in (0, \infty)$ such that

$$(5.12) \quad at \in (n + \log_2 c_1 + \varepsilon/q_-, \infty).$$

Let $P \subset \mathbb{R}^n$ be a given dyadic cube. Next we show that

$$(5.13) \quad \frac{1}{[\phi(P)]^t} \left\| \left\{ [\varphi_j^{*,a}(2^{js(\cdot)}f)]^t \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)/t}(L^{p(\cdot)/t}(P))} \lesssim 1$$

with implicit positive constant independent of P , which, by Lemma 2.3 and Remark 2.9(i), is equivalent to prove that $\sum_{j=(j_P \vee 0)}^\infty I_{P,j} \lesssim 1$, where

$$I_{P,j} := \inf \left\{ \lambda_j \in (0, \infty) : \varrho_{\frac{p(\cdot)}{t}} \left(\frac{c\chi_P[\varphi_j^{*,a}(2^{js(\cdot)}f)]^t}{[\phi(P)]^t \lambda_j^{t/q(\cdot)}} \right) \leq 1 \right\}$$

with c being a positive constant sufficiently small. Since

$$\left[\varphi_j^{*,a}(2^{js(\cdot)}f)(x) \right]^t = \sup_{y \in \mathbb{R}^n} \frac{2^{js(y)t} |\varphi_j * f(y)|^t}{(1 + 2^j|x-y|)^{at}},$$

it follows, from Lemma 5.4, that, for all $j \in \mathbb{Z}_+ \cap [(j_P \vee 0), \infty)$,

$$\begin{aligned} I_{P,j} &\leq \sum_{k=1}^\infty \inf \left\{ \eta_j \in (0, \infty) : \varrho_{\frac{p(\cdot)}{t}} \left(\frac{\chi_{P_k^n} 2^{js(\cdot)t} |\varphi_j * f|^t}{2^{k\varepsilon t/q(\cdot)} [\phi(P_k^n)]^t \eta_j^{t/q(\cdot)}} \right) \leq 1 \right\} + 2^{-\tilde{\sigma}[j-(j_P \vee 0)]} \\ &= \sum_{k=1}^\infty 2^{-k\varepsilon} \inf \left\{ \eta_j \in (0, \infty) : \varrho_{\frac{p(\cdot)}{t}} \left(\frac{\chi_{P_k^n} 2^{js(\cdot)t} |\varphi_j * f|^t}{[\phi(P_k^n)]^t \eta_j^{t/q(\cdot)}} \right) \leq 1 \right\} + 2^{-\tilde{\sigma}[j-(j_P \vee 0)]} =: \widetilde{\delta}_j^P, \end{aligned}$$

where $P_k^n := 2^{k+1+n}P$ and $\tilde{\sigma} \in (0, \frac{at-n}{4(1/q_- - 1/q_+)})$. From this, we further deduce that

$$\begin{aligned} \sum_{j=(j_P \vee 0)}^\infty I_{P,j} &\lesssim \sum_{k=1}^\infty \frac{2^{-k\varepsilon}}{[\phi(P_k^n)]^t} \left\| \left\{ 2^{js(\cdot)} |\varphi_j * f| \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P_k^n))}^t + 1 \\ &\lesssim \sum_{k=1}^\infty 2^{-k\varepsilon} \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}^t + 1 \lesssim 1, \end{aligned}$$

which implies that (5.13) holds true. This finishes the proof of Theorem 5.1. \square

As applications of Theorem 5.1, we obtain more equivalent quasi-norms of Besov-type spaces with variable smoothness and integrability. To this end, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\left\| f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \right\|_1 := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} |\varphi_j * f| \right\}_{j \geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}$$

and

$$\left\| f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \right\|_2 := \sup_{Q \in \mathcal{Q}^*} \sup_{x \in Q} [\phi(Q)]^{-1} |Q|^{-s(x)/n} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} |\varphi_{j_Q} * f(x)|.$$

Theorem 5.5. *Let p, q, s, ϕ be as in Definition 2.12.*

(i) *Assume that $p_+ \in (0, \infty)$ and $c_1 \in (0, 2^{n/p_+})$. Then $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_1 < \infty$; moreover, there exists a positive constant C , independent of f , such that*

$$(5.14) \quad \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \left\| f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \right\|_1 \leq C \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

(ii) *Assume that $p_- \in (0, \infty)$ and $c_1 \in (0, 2^{-n/p_-})$. Then $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_2 < \infty$; moreover, there exists a positive constant C , independent of f , such that*

$$(5.15) \quad C^{-1} \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \left\| f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n) \right\|_2 \leq C \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

Proof. Let $P \subset \mathbb{R}^n$ be a given dyadic cube and, for all $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $f_j(x) := 2^{js(x)} |\varphi_j * f(x)|$.

We first prove (i). Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_1 < \infty$. Then, by definitions, we easily find that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq \|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_1$ and hence $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Conversely, let $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$. To complete the proof of (i), it suffices to show the second inequality of (5.14).

When $q_+ \in (0, \infty)$, by Remark 2.9(iv), we have

$$(5.16) \quad \frac{1}{\phi(P)} \left\| \{f_j\}_{j \geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ \lesssim \frac{1}{\phi(P)} \left\| \{f_j\}_{j=0}^{(j_P \vee 0)-1} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} + \frac{1}{\phi(P)} \left\| \{f_j\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} =: I_{P,1} + I_{P,2},$$

where $I_{P,1} = 0$ if $j_P \leq 0$. Obviously, $I_{P,2} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}$. To estimate $I_{P,1}$, without loss of generality, we may assume that $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} = 1$ and show that $I_{P,1} \lesssim 1$ in the case that $j_P > 0$. Observe that, for all $j \in \mathbb{Z}_+$ with $j \leq j_P - 1$, there exists a unique dyadic cube P_j such that $P \subset P_j$ and $\ell(P_j) = 2^{-j}$. It follows that, for all $x \in P$,

$$(5.17) \quad f_j(x) := 2^{js(x)} |\varphi_j * f(x)| \lesssim \inf_{y \in P_j} \varphi_j^{*,a}(2^{js(\cdot)} f)(y)$$

and, moreover,

$$(5.18) \quad \left\| [\phi(P)]^{-1} \chi_P f_j \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| [\phi(P)]^{-1} \chi_P \inf_{y \in P_j} \varphi_j^{*,a}(2^{js(\cdot)} f)(y) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ \lesssim \frac{\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [\phi(P)]^{-1} \|\varphi_j^{*,a}(2^{js(\cdot)} f)\|_{L^{p(\cdot)}(P_j)}$$

$$\lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \frac{\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)} \phi(P_j)}{\|\chi_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \phi(P)},$$

where we used Theorem 5.1 in the last inequality. On the other hand, by [80, Lemma 2.6], we find that

$$\|\chi_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \gtrsim 2^{-j\frac{n}{p_+}} 2^{jP\frac{n}{p_+}} \|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

and, by (2.4) and (2.5), we see that $\phi(P) \gtrsim 2^{j\log_2 c_1} 2^{-jP\log_2 c_1} \phi(c_{P_j}, 2^{-j})$. Thus, by (5.18), we further conclude that

$$(5.19) \quad \|[\phi(P)]^{-1} \chi_P f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} 2^{(j-j_P)(\frac{n}{p_+} - \log_2 c_1)},$$

which, together with (i) and (ii) of Lemma 3.14, implies that

$$\begin{aligned} & \inf \left\{ \lambda_j : \varrho_{p(\cdot)} \left([\phi(P)]^{-1} \lambda_j^{-1/q(\cdot)} \chi_P f_j \right) \leq 1 \right\} \\ & \lesssim \|[\phi(P)]^{-1} \chi_P f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-} + \|[\phi(P)]^{-1} \chi_P f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_+} \\ & \lesssim 2^{(j-j_P)(\frac{n}{p_+} - \log_2 c_1)q_-} + 2^{(j-j_P)(\frac{n}{p_+} - \log_2 c_1)q_+}. \end{aligned}$$

From this and $c_1 \in (0, 2^{n/p_+})$, we deduce that there exists a positive constant C_0 such that

$$\sum_{j=0}^{j_P-1} \inf \left\{ \lambda_j : \varrho_{p(\cdot)} \left(\frac{\chi_P f_j}{C_0 \phi(P) \lambda_j^{1/q(\cdot)}} \right) \leq 1 \right\} \leq 1,$$

namely, $\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{[C_0 \phi(P)]^{-1} \chi_P f_j\}_{j=0}^{j_P-1}) \leq 1$, which, combined with Remark 2.9(i), implies that $I_{P,1} \lesssim 1$. Therefore, by (5.16), we find that

$$\|f|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}\|_1 \lesssim \sup_{P \in \mathcal{Q}} (I_{P,1} + I_{P,2}) \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)},$$

which completes the proof of the second inequality of (5.14) in the case $q_+ \in (0, \infty)$.

We now consider the case that $q_+ = \infty$. In this case, $q \equiv \infty$ by Remark 2.10(iii). From (5.19) and $c_1 \in (0, 2^{n/p_+})$, we deduce that, for $j_P \in \mathbb{N}$,

$$\sup_{j \in \mathbb{Z}_+, j \leq j_P} [\phi(P)]^{-1} \|f_j\|_{L^{p(\cdot)}(P)} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \sup_{j \in \mathbb{Z}_+, j \leq j_P} 2^{(j-j_P)(\frac{n}{p_+} - \log_2 c_1)} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

By this, we know that

$$\begin{aligned} \|f|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)}\|_1 & \lesssim \sup_{P \in \mathcal{Q}} \left\{ \sup_{j \in \mathbb{Z}_+, j \leq (j_P \vee 0)} \frac{\|f_j\|_{L^{p(\cdot)}(P)}}{\phi(P)} + \sup_{j \in \mathbb{Z}_+, j \geq (j_P \vee 0)} \frac{\|f_j\|_{L^{p(\cdot)}(P)}}{\phi(P)} \right\} \\ & \lesssim \|f\|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of the second inequality of (5.14) in the case that $q_+ = \infty$ and hence (i) of Theorem 5.5.

Next, we show (ii). Let $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$. On the other hand, for all $Q \in \mathcal{Q}^*$ and $x \in Q$, by Theorem 5.1 and (5.17), we easily see that

$$\frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\phi(Q)} f_{j_Q}(x) \lesssim \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\phi(Q)} \inf_{y \in Q} \varphi_{j_Q}^{*,a}(2^{j_Q s(\cdot)} f)(y)$$

$$\lesssim [\phi(Q)]^{-1} \|\varphi_{jQ}^{*,a}(2^{jQ} s^{(\cdot)} f)\|_{L^{p(\cdot)}(Q)} \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.$$

This implies that $\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_2 \lesssim \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} < \infty$.

Conversely, let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_2 < \infty$. We need to show the first inequality of (5.15). To this end, for all $j \geq (j_P \vee 0)$ and $x \in \mathbb{R}^n$, $\mathcal{Q}_{P,j}^* := \{Q \in \mathcal{Q}^* : Q \subset P, \ell(Q) = 2^{-j}\}$ and, for all $Q \in \mathcal{Q}_{P,j}^*$, let

$$g(Q, P)(x) := [\phi(P)]^{-1} \phi(Q) \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \chi_Q(x).$$

When $q_+ \in (0, \infty)$, by [80, Lemma 2.6], (2.4) and (2.5), we find that

$$\left\| \sum_{Q \in \mathcal{Q}_{P,j}^*} g(Q, P) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 2^{(j-j_P)(\log_2 c_1 + \frac{n}{p_-})},$$

which, combined with (i) and (ii) of Lemma 3.14, implies that

$$\begin{aligned} & \varrho_{\ell q(\cdot)}(L^{p(\cdot)}) \left(\left\{ \sum_{Q \in \mathcal{Q}_{P,j}^*} g(Q, P) \right\} \right) \\ &= \sum_{j=(j_P \vee 0)}^{\infty} \inf \left\{ \lambda_j \in (0, \infty) : \varrho_{p(\cdot)} \left(\sum_{Q \in \mathcal{Q}_{P,j}^*} \frac{g(Q, P)}{\lambda_j^{1/q(\cdot)}} \right) \leq 1 \right\} \\ &\leq \sum_{j=(j_P \vee 0)}^{\infty} \left[\left\| \sum_{Q \in \mathcal{Q}_{P,j}^*} g(Q, P) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_-} + \left\| \sum_{Q \in \mathcal{Q}_{P,j}^*} g(Q, P) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{q_+} \right] \\ &\lesssim \sum_{j=(j_P \vee 0)}^{\infty} \left[2^{(j-j_P)(\log_2 c_1 + \frac{n}{p_-})q_-} + 2^{(j-j_P)(\log_2 c_1 + \frac{n}{p_-})q_+} \right] \lesssim 1. \end{aligned}$$

By this and Remark 2.9(i), we conclude that

$$\left\| \left\{ \sum_{Q \in \mathcal{Q}_{P,j}^*} g(Q, P) \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell q(\cdot)(L^{p(\cdot)}(P))} \lesssim 1.$$

Therefore,

$$\begin{aligned} & \left\| \{[\phi(P)]^{-1} \chi_P f_j\}_{j \geq (j_P \vee 0)} \right\|_{\ell q(\cdot)(L^{p(\cdot)}(P))} \\ &\lesssim \left\| \left\{ \frac{1}{\phi(P)} \sum_{Q \in \mathcal{Q}_{P,j}^*} \chi_Q f_j \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell q(\cdot)(L^{p(\cdot)}(P))} \\ &\lesssim \|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_2 \left\| \left\{ \sum_{Q \in \mathcal{Q}_{P,j}^*} g(Q, P) \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell q(\cdot)(L^{p(\cdot)}(P))} \lesssim \|f|B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)\|_2, \end{aligned}$$

which implies that the first inequality of (5.15) holds true in the case that $q_+ \in (0, \infty)$. The proof of the case that $q_+ = \infty$ is similar and more simple, the details being omitted. This finishes the proof of (ii) and hence Theorem 5.5. \square

As another application of Theorem 5.1, we obtain the following conclusion.

Proposition 5.6. *Let p , q , s and ϕ be as in Definition 2.12. Then*

$$(5.20) \quad \mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Proof. By Proposition 4.1, we see that $B_{p(\cdot), q_-}^{s(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)$. Thus, to prove (5.20), it suffices to show that

$$(5.21) \quad \mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), q_-}^{s(\cdot), \phi}(\mathbb{R}^n) \quad \text{and} \quad B_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

The first embedding of (5.21) can be obtained by an argument similar to that used in the proof of [76, Proposition 3.20], the details being omitted. Next we give the proof of the second one. To this end, we only need to show that there exists an $M \in \mathbb{N}$ such that, for all $f \in B_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)$ and $h \in \mathcal{S}(\mathbb{R}^n)$, $|\langle f, h \rangle| \lesssim \|h\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)} \|f\|_{B_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)}$.

Let φ , ψ , Φ and Ψ be as in (3.1). Then, by the Calderón reproducing formula in [78, Lemma 2.3], together with [78, Lemma 2.4], we find that

$$(5.22) \quad \begin{aligned} |\langle f, h \rangle| &\leq \int_{\mathbb{R}^n} |\Phi * f(x)| |\Psi * h(x)| dx + \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\varphi_j * f(x)| |\psi_j * h(x)| dx \\ &\lesssim \|h\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)} \sum_{j=0}^{\infty} 2^{-jM} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} |\varphi_j * f(x)| (1 + |x|)^{-(n+M)} dx, \end{aligned}$$

where we used φ_0 to replace Φ . Notice that, for any $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}^n$, $a \in (0, \infty)$ and $y \in Q_{jk}$,

$$\begin{aligned} \int_{Q_{0k}} |\varphi_j * f(x)| dx &\lesssim \varphi_j^{*, a}(2^{js(\cdot)} f)(y) \int_{Q_{0k}} 2^{-js(x)} (1 + 2^j|x| + 2^j|y|)^a dx \\ &\lesssim 2^{-js_-} \varphi_j^{*, a}(2^{js(\cdot)} f)(y) 2^{ja} (1 + |k|)^a. \end{aligned}$$

It follows that

$$\int_{Q_{0k}} |\varphi_j * f(x)| dx \lesssim 2^{j(a-s_-)} (1 + |k|)^a \inf_{y \in Q_{jk}} \varphi_j^{*, a}(2^{js(\cdot)} f)(y),$$

which, combined with (5.22), Theorem 5.1, Lemmas 3.6 and 3.7, implies that

$$\begin{aligned} |\langle f, h \rangle| &\lesssim \|h\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)} \sum_{j=0}^{\infty} 2^{-j(M+s_- - a)} \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{a-n-M} \frac{\|\varphi_j^{*, a}(2^{js(\cdot)} f)\|_{L^{p(\cdot)}(Q_{jk})}}{\|\chi_{Q_{jk}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\lesssim \|h\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)} \|f\|_{B_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \frac{2^{-j(M+s_- - a)}}{(1 + |k|)^{M+n-a}} \frac{\phi(Q_{jk})}{\|\chi_{Q_{jk}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\lesssim \|h\|_{\mathcal{S}_{M+1}(\mathbb{R}^n)} \|f\|_{B_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)}, \end{aligned}$$

where a is as in Theorem 5.1 and M is large enough. This finishes the proof of Proposition 5.6. \square

Remark 5.7. (i) When $\phi \equiv 1$, Proposition 5.6 was proved in [3, Theorem 6.10].

(ii) When p, q, s and ϕ are as in Remark 2.10(ii), Proposition 5.6 was obtained in [78, Proposition 2.3].

Next we establish the atomic characterization of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$.

Definition 5.8. Let $k \in \mathbb{Z}_+$ and $L \in \mathbb{Z}$. A measurable function a_Q on \mathbb{R}^n is called a (K, L) -smooth atom supported near $Q := Q_{jk} \in \mathcal{Q}$ if it satisfies the following conditions:

(A1) (*support condition*) $\text{supp } a_Q \subset 3Q$;

(A2) (*vanishing moment*) when $j \in \mathbb{N}$, $\int_{\mathbb{R}^n} x^\gamma a_Q(x) dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| < L$;

(A3) (*smoothness condition*) for all multi-indices $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq K$, $|D^\alpha a_Q(x)| \leq 2^{(|\alpha|+n/2)j}$.

A collection $\{a_Q\}_{Q \in \mathcal{Q}^*}$ is called a *family of (K, L) smoothness atoms*, if each a_Q is a (K, L) -smooth atom supported near Q .

We point out that, if $L \leq 0$, then the vanishing moment condition (A2) is avoid.

Theorem 5.9. Let p, q, s and ϕ be as in Definition 2.12.

(i) Let $K \in (s_+ + \log_2 c_1, \infty)$ and

$$(5.23) \quad L \in (n/\min\{1, p_-\} - n - s_-, \infty).$$

Suppose that $\{a_Q\}_{Q \in \mathcal{Q}^*}$ is a family of (K, L) -smooth atoms and $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \in b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$. Then $f := \sum_{Q \in \mathcal{Q}^*} t_Q a_Q$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and $\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq C \|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}$ with C being a positive constant independent of t .

(ii) Conversely, if $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$, then, for any given $K, L \in \mathbb{Z}_+$, there exist sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ and $\{a_Q\}_{Q \in \mathcal{Q}^*}$ of (K, L) -smooth atoms such that $f = \sum_{Q \in \mathcal{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|t\|_{b_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} \leq C \|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}$ with C being a positive constant independent of f .

Remark 5.10. (i) Even when $\phi \equiv 1$, conclusions of Theorem 5.9 cover [17, Theorem 3], in which the case that $q_+ = \infty$ is not included.

(ii) In the case that p, q, s and ϕ are as in Remark 2.13(ii), Theorem 5.9 was proved in [17, Theorem 3] and partly obtained in [78, Theorem 3.3].

(iii) A sequence $\{a_Q\}_{Q \in \mathcal{Q}^*}$ is called a *family of smooth atoms of $B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)$* if, for each $Q \in \mathcal{Q}^*$, a_Q is a (K, L) -smooth atom with K and L as in Theorem 5.9(i).

To prove Theorem 5.9, we need the following two technical lemmas. The first one was proved in [21, Lemma 3.3] and the second one is a Hardy-type inequality which is just [18, Lemma 3.11].

Lemma 5.11. Let $\{\varphi_j\}_{j \in \mathbb{Z}_+}$ be as in Definition 2.12 and $a_{Q_{vk}}$ with $v \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$ be a (K, L) -smooth atom. Then, for all $M \in (0, \infty)$, there exist positive constants C_1 and C_2 such that, for all $x \in \mathbb{R}^n$, when $j \leq v$,

$$|\varphi_j * a_{Q_{vk}}(x)| \leq C_1 2^{vn/2} 2^{-(v-j)(L+n)} (1 + 2^j |x - x_{Q_{vk}}|)^{-M}$$

and, when $j > v$,

$$|\varphi_j * a_{Q_{vk}}(x)| \leq C_2 2^{vn/2} 2^{-(j-v)K} (1 + 2^v |x - x_{Q_{vk}}|)^{-M}.$$

Lemma 5.12. Let $a \in (0, 1)$, $J \in \mathbb{Z}$, $q \in (0, \infty]$ and $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$ be a sequence of positive real numbers. For all $k \in [J \vee 0, \infty)$, let $\delta_k := \sum_{j=(J \vee 0)}^k a^{k-j} \varepsilon_j$ and $\eta_k := \sum_{j=k}^\infty a^{j-k} \varepsilon_j$. Then there exists a positive constant C , depending only on a and q , such that

$$\left(\sum_{k=(J \vee 0)}^\infty \delta_k^q \right)^{1/q} + \left(\sum_{k=(J \vee 0)}^\infty \eta_k^q \right)^{1/q} \leq C \left(\sum_{k=(J \vee 0)}^\infty \varepsilon_k^q \right)^{1/q}.$$

Proof of Theorem 5.9. The proof of (ii) is similar to that of [78, Theorem 3.3] (see also [22, Theorem 4.1]). Indeed, by repeating the argument that used in the proof of [78, Theorem 3.3], with [78, Lemma 2.8] therein replaced by Lemma 3.15, we can prove (ii), the details being omitted.

Next we prove (i) by two steps. First, we show that $f := \sum_{Q \in \mathcal{Q}^*} t_Q a_Q$ converges in $\mathcal{S}'(\mathbb{R}^n)$. To this end, it suffices to prove that

$$(5.24) \quad \lim_{N \rightarrow \infty, \Lambda \rightarrow \infty} \sum_{j=0}^N \sum_{k \in \mathbb{Z}^n, |k| \leq \Lambda} t_{Q_{jk}} a_{Q_{jk}}$$

exists in $\mathcal{S}'(\mathbb{R}^n)$. By (5.23), we find that there exists $r \in (0, \min\{1, p_-\})$ such that $s_- + \frac{n}{p_-}(r-1) > -L$. Let, for all $x \in \mathbb{R}^n$, $\tilde{p}(x) := p(x)/r$ and \tilde{s} be a measurable function on \mathbb{R}^n such that $s(x) - \frac{n}{p(x)} = \tilde{s}(x) - \frac{n}{\tilde{p}(x)}$. Then $\tilde{s}_- \geq s_- + \frac{n}{p_-}(r-1) > -L$. Therefore, by Proposition 4.2 and an argument similar to that used in the proof of [76, Theorem 3.8], we conclude that there exist $\delta_0 \in (\log_2 c_1, \infty)$, $a \in (n, \infty)$, $c_0 \in \mathbb{N}$ and $R \in (0, \infty)$ being large enough such that, for all $h \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}_+$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^n, |k| \leq \Lambda} t_{Q_{jk}} a_{Q_{jk}}(y) h(y) dy \right| \\ & \lesssim 2^{-j(L+\tilde{s}_-)} \sum_{v=0}^{\infty} 2^{-v\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-L-a)} \left\| \sum_{k \in \mathbb{Z}^n} 2^{j\tilde{s}(\cdot)} |t_{Q_{jk}}| \tilde{\chi}_{Q_{jk}} \right\|_{L^{\tilde{p}(\cdot)}(Q(0, 2^{i+v+c_0}))} \\ & \lesssim 2^{-j(L+\tilde{s}_-)} \sum_{v=0}^{\infty} 2^{-v\delta_0} \sum_{i=0}^{\infty} 2^{-i(R-L-a)} \phi(Q(0, 2^{i+v+c_0})) \|t\|_{b_{\tilde{p}(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)} \\ & \lesssim 2^{-j(L+\tilde{s}_-)} \sum_{v=0}^{\infty} 2^{-v(\delta_0 - \log_2 c_1)} \sum_{i=0}^{\infty} 2^{-i(R-L-a-\log_2 c_1)} \|t\|_{b_{p(\cdot), \infty}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim 2^{-j(L+\tilde{s}_-)} \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}. \end{aligned}$$

By this and the fact that $L > -\tilde{s}_-$, we find that the limit of (5.24) exists in $\mathcal{S}'(\mathbb{R}^n)$.

Second, we prove that

$$(5.25) \quad \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim \|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

Without loss of generality, we may assume that $\|t\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} = 1$ and show $\|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim 1$.

Case I $q_+ \in (0, \infty)$. By Remark 3.2, we see that, for all $R \in \mathcal{D}_0(\mathbb{R}^n)$,

$$(5.26) \quad \frac{1}{\phi(R)} \left\| \left\{ \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-v}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\}_{v \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(R))} \lesssim 1$$

with implicit positive constant independent of R . Since $f = \sum_{Q \in \mathcal{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$, it follows that, for all $P \in \mathcal{Q}$,

$$\varphi_j * f = \left\{ \sum_{v=0}^{(j_P \vee 0)-1} + \sum_{v=(j_P \vee 0)}^j + \sum_{v=j+1}^{\infty} \right\} \sum_{\ell(Q)=2^{-v}} t_Q \varphi_j * a_Q =: S_{j,1} + S_{j,2} + S_{j,3},$$

where $\sum_{v=0}^{(j_P \vee 0)-1} \dots = 0$ if $j_P \leq 0$. Thus, by Remark 2.9(iv), we find that

$$I_P := \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} \varphi_j * f \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}$$

$$\lesssim \sum_{i=1}^3 \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} S_{j,i} \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} =: I_{P,1} + I_{P,2} + I_{P,3}.$$

In what follows, let $r \in (0, \min\{1, p_-\})$ satisfy $L + n - n/r + s_- > 0$.

We show that $I_{P,1} \lesssim 1$. To this end, it suffices to consider the case that $j_P > 0$ and prove that there exists a positive constant C such that

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\left\{ \frac{\chi_P 2^{js(\cdot)}}{C\phi(P)} \sum_{v=0}^{j_P-1} \sum_{\ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q| \right\}_{j \geq j_P} \right) \leq 1,$$

which, by Remark 2.9(ii), is equivalent to show that

$$J_{P,1} := \sum_{j=j_P}^{\infty} \left\| \left[\frac{\chi_P}{\phi(P)} 2^{js(\cdot)} \sum_{v=0}^{j_P-1} \sum_{\ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q| \right]^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \lesssim 1.$$

By Lemma 5.11 and (4.1), we find that

$$(5.27) \quad J_{P,1} \lesssim \sum_{j=j_P}^{\infty} \left\| \left[\frac{\chi_P}{\{\phi(P)\}^r} 2^{js(\cdot)r} \sum_{v=0}^{j_P-1} \sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}|^r |Q_{vk}|^{-r/2} \right. \right. \\ \left. \left. \times 2^{(v-j)Kr} (1 + 2^v |\cdot - x_{Q_{vk}}|)^{-Mr} \right]^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)}^{\frac{1}{r}},$$

where $M \in (0, \infty)$ is large enough. On the other hand, by the proof of [76, Theorem 3.8(i)], we know that, for all $v, j \in \mathbb{Z}_+$ with $v \leq j$ and $x \in P$,

$$2^{js(x)r} \sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}|^r |Q_{vk}|^{-\frac{r}{2}} 2^{(v-j)Kr} (1 + 2^v |x - x_{Q_{vk}}|)^{-Mr} \\ \lesssim 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{C_{\log}(s)}{r})r} \eta_{v,ar} * \left(\left[\sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}| 2^{vs(\cdot)} \tilde{\chi}_{Q_{vk}} \chi_{Q(c_P, 2^{-v+c_0})} \right]^r \right) (x),$$

where $a \in (n/r, \infty)$, c_P is the center of P and $c_0 \in \mathbb{N}$ independent of x, P, i, v and k . From this, (5.27), and (i) and (ii) of Lemma 3.14, we deduce that $J_{P,1} \lesssim \sum_{j=j_P}^{\infty} [(J_{P,1}^j)^{q_-} + (J_{P,1}^j)^{q_+}]$, where

$$J_{P,1}^j := \left\| \frac{\chi_P}{[\phi(P)]^r} \sum_{v=0}^{j_P-1} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-C_{\log}(s)/r)r} \right. \\ \left. \times \eta_{v,ar} * \left(\sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}|^r 2^{vs(\cdot)r} |Q_{vk}|^{-\frac{r}{2}} \chi_{Q_{vk}} \chi_{Q(c_P, 2^{-v+c_0})} \right) \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}^{\frac{1}{r}}.$$

By Remark 2.7(i), (5.26), Remarks 2.10(i) and 3.2, we find that

$$J_{P,1}^j \lesssim \left\{ \frac{1}{[\phi(P)]^r} \sum_{v=0}^{j_P-1} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-C_{\log}(s)/r)r} \right.$$

$$\begin{aligned} & \times \left\| \sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}|^r 2^{vs(\cdot)r} |Q_{vk}|^{-\frac{r}{2}} \chi_{Q_{vk}} \chi_{Q(c_P, 2^{i-v+c_0})} \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}^{\frac{1}{r}} \\ & \lesssim \left\{ \sum_{v=0}^{j_P-1} 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-C_{\log}(s)/r)r} \frac{[\phi(Q(c_P, 2^{i-v+c_0}))]^r}{[\phi(P)]^r} \right\}^{\frac{1}{r}}. \end{aligned}$$

By this, (2.4) and the fact that $K \in (s_+ + \log_2 c_1, \infty)$, we know that

$$\sum_{j=j_P}^{\infty} (J_{P,1}^j)^{q_-} \lesssim 2^{j_P q_- \log_2 c_1} \sum_{j=j_P}^{\infty} \left\{ 2^{j(s_+-K)} \sum_{v=0}^{j_P-1} 2^{(K-s_+-\log_2 c_1)vr} \right\}^{\frac{q_-}{r}} \lesssim 1$$

and $\sum_{j=j_P}^{\infty} (J_{P,1}^j)^{q_+} \lesssim 1$, where M is chosen large enough such that $M > a + C_{\log}(s)/r + \log_2 c_1$, which implies $I_{P,1} \lesssim 1$. This is a desired estimate.

We now estimate $I_{P,2}$. By Lemma 5.11, we see that, for all $M \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\sum_{k \in \mathbb{Z}^n} 2^{js(x)} |t_{Q_{vk}}| |\varphi_j * a_{Q_{vk}}(x)| \lesssim 2^{(v-j)(K-s_+)} \sum_{k \in \mathbb{Z}^n} |Q_{vk}|^{-\frac{s(x)}{n} + \frac{1}{2}} |t_{Q_{vk}}| (1 + 2^v |x - x_{Q_{vk}}|)^{-M}$$

and hence, for all $r \in (0, \min\{1/q_+, p_-/q_+\})$,

$$\begin{aligned} (5.28) \quad & \left\| \left[\frac{\chi_P}{\phi(P)} 2^{js(\cdot)} S_{j,2} \right]^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}^{q(\cdot)} \\ & \lesssim \left\{ \sum_{v=(j_P \vee 0)}^j 2^{(v-j)q_-(K-s_+)} \left\| \left[\frac{\chi_P}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{|Q|^{-\frac{s(\cdot)}{n} + \frac{1}{2}} |t_Q|}{(1 + 2^v |\cdot - x_Q|)^M} \right]^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)} \right\}^{\frac{1}{r}}. \end{aligned}$$

We claim that there exists a positive constant c such that

$$\begin{aligned} (5.29) \quad & \left\| \left[\frac{c\chi_P}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{|Q|^{-\frac{s(\cdot)}{n} + \frac{1}{2}} |t_Q|}{(1 + 2^v |\cdot - x_Q|)^M} \right]^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)}^{rq(\cdot)} \\ & \leq \sum_{i=0}^{\infty} 2^{-i\tau} \left\| \left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-v}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right]^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)}^{rq(\cdot)} + 2^{-v} =: \delta_v^P, \end{aligned}$$

where $Q_i^0 := Q(c_P, 2^{i-j_P+c_0})$ with some $c_0 \in \mathbb{N}$ and $\tau \in (0, \infty)$.

From the above claim, (5.28), Lemma 5.12, the Minkowski inequality and (5.26), we deduce that

$$\begin{aligned} & \sum_{j=(j_P \vee 0)}^{\infty} \left\| \left[\frac{\chi_P}{\phi(P)} 2^{js(\cdot)} S_{j,2} \right]^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)}^{q(\cdot)} \\ & \lesssim \sum_{j=(j_P \vee 0)}^{\infty} \left\{ \sum_{i=0}^{\infty} 2^{-i\tau} \left\| \left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-j}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right]^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)} \right\}^{\frac{1}{r}} + \sum_{j=0}^{\infty} 2^{-j/r} \end{aligned}$$

$$\begin{aligned}
&\lesssim \left\{ \sum_{i=0}^{\infty} 2^{-i\tau} \left(\sum_{j=(j_P \vee 0)}^{\infty} \left\| \left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-j}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right]^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(\mathbb{R}^n)} \right)^r \right\}^{\frac{1}{r}} + 1 \\
&\lesssim \left\{ \sum_{i=0}^{\infty} 2^{-i\tau} \right\}^{1/r} + 1 \lesssim 1,
\end{aligned}$$

which, together with Lemma 2.3 and Remark 2.9(i) again, implies that $I_{P,2} \lesssim 1$.

Let us prove (5.29) now. Obviously, it suffices to show that

$$\left\| [\delta_v^P]^{-1} \left[\frac{c\chi_P}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{|Q|^{-\frac{s(\cdot)}{n} + \frac{1}{2}} |t_Q|}{(1 + 2^v |\cdot - x_Q|)^M} \right]^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)} \leq 1,$$

which, via Lemma 3.14, is a consequence of

$$\mathcal{A} := \left\| [\delta_v^P]^{-\frac{1}{rq(\cdot)}} \frac{\chi_P}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{|Q|^{-\frac{s(\cdot)}{n} + \frac{1}{2}} |t_Q|}{(1 + 2^v |\cdot - x_Q|)^M} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim 1.$$

Taking $t \in (0, \min\{1, p_-\})$ and using some arguments similar to those used in [18, pp. 29-30], we conclude that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
(5.30) \quad &\sum_{\ell(Q)=2^{-v}} [\delta_v^P]^{-\frac{1}{rq(x)}} \frac{\chi_P(x)}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{|Q|^{-\frac{s(x)}{n} + \frac{1}{2}} |t_Q|}{(1 + 2^v |x - x_Q|)^M} \\
&\lesssim \sum_{i=0}^{(v-c_0) \vee 0} 2^{i\zeta} \left\{ \mathcal{M} \left(\left[\frac{\chi_{Q_i^0}}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{\frac{1}{rq(\cdot)}}} |t_Q| \tilde{\chi}_Q \right]^t \right) \right\}^{1/t} + \sum_{i=(v-c_0) \vee 0}^{\infty} 2^{i\vartheta} \dots,
\end{aligned}$$

where $\zeta := -M + \frac{n}{t} + \frac{2}{r} C_{\log}(q) + C_{\log}(s)$ and $\vartheta := -M + \frac{n}{t} + \frac{2}{r} (\frac{1}{q_-} - \frac{1}{q_+}) + s_+ - s_-$. Taking M large enough such that

$$M > \max \left\{ \frac{n}{t} + \frac{2}{r} C_{\log}(q) + C_{\log}(s) + \log_2 c_1, \frac{2}{r} \left(\frac{1}{q_-} - \frac{1}{q_+} \right) + s_+ - s_- + \log_2 c_1 \right\} + \tau,$$

then, by (2.4), Lemmas 5.3 and 3.14, we know that

$$\begin{aligned}
&\sum_{i=0}^{(v-c_0) \vee 0} 2^{i\zeta t} \left\| \mathcal{M} \left(\left[\frac{\chi_{Q_i^0}}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{\frac{1}{rq(\cdot)}}} |t_Q| \tilde{\chi}_Q \right]^t \right) \right\|_{L^{\frac{p(\cdot)}{t}}(\mathbb{R}^n)} \\
&\lesssim \sum_{i=0}^{(v-c_0) \vee 0} 2^{it(\zeta + \log_2 c_1)} \left\| \frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-v}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{\frac{1}{rq(\cdot)}}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^t \\
&\lesssim \sum_{i=0}^{(v-c_0) \vee 0} 2^{it(\zeta + \log_2 c_1)} \left\| \frac{1}{\delta_v^P} \left[\frac{\chi_{Q_i^0}}{\phi(Q_i^0)} \sum_{\ell(Q)=2^{-v}} 2^{vs(\cdot)} |t_Q| \tilde{\chi}_Q \right]^{rq(\cdot)} \right\|_{L^{\frac{p(\cdot)}{rq(\cdot)}}(\mathbb{R}^n)}^{\frac{t}{rq_+}} \lesssim \sum_{i=0}^{\infty} 2^{it(\zeta + \log_2 c_1 + \tau)} \lesssim 1,
\end{aligned}$$

where we used the definition of δ_v^P in the penultimate inequality and, similarly,

$$\sum_{i=(v-c_0) \vee 0}^{\infty} 2^{i\vartheta t} \left\| \mathcal{M} \left(\left[\frac{\chi_{Q_i^0}}{\phi(P)} \sum_{\ell(Q)=2^{-v}} \frac{2^{vs(\cdot)}}{[\delta_v^P]^{\frac{1}{r q(\cdot)}}} |t_Q| \tilde{\chi}_Q \right]^t \right) \right\|_{L^{\frac{p(\cdot)}{t}}(\mathbb{R}^n)} \lesssim 1.$$

From this and (5.30), we deduce that $\mathcal{A} \lesssim 1$, which implies that (5.29) holds true and then completes the proof that $I_{P,2} \lesssim 1$.

We next prove that $I_{P,3} \lesssim 1$. To this end, it suffices to show that

$$\varrho_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \left(\left\{ \frac{\chi_P}{\tilde{C}\phi(P)} 2^{js(\cdot)} \sum_{v=(j_P \vee 0)}^{\infty} \sum_{\ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q| \right\}_{j \geq (j_P \vee 0)}^r \right) \leq 1$$

for some positive constant \tilde{C} large enough independent of P , which, by Definition 2.8, is equivalent to show that $\sum_{j=(j_P \vee 0)}^{\infty} Y_j^P \lesssim 1$, where, for all $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

$$Y_j^P := \inf \left\{ \lambda_j \in (0, \infty) : \varrho_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \left(\frac{\chi_P [2^{js(\cdot)} \sum_{v=j}^{\infty} \sum_{\ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q|]^r}{\tilde{C}[\phi(P)\lambda_j^{1/q(\cdot)}]^r} \right) \leq 1 \right\}.$$

We claim that, for all $P \in \mathcal{Q}$ and $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

$$\begin{aligned} (5.31) \quad Y_j^P &\leq 2^{-j} + \sum_{v=j}^{\infty} 2^{(j-v)d} \sum_{i=0}^{\infty} 2^{-i\tilde{d}} \\ &\quad \times \inf \left\{ \xi_v \in (0, \infty) : \varrho_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \left(\frac{\chi_{P_i} [\sum_{\ell(Q)=2^{-v}} |t_Q| 2^{-vs(\cdot)} \chi_Q]^r}{[\phi(P_i)\xi_v^{1/q(\cdot)}]^r} \right) \leq 1 \right\} \\ &=: 2^{-j} + \sum_{v=j}^{\infty} 2^{(j-v)d} \sum_{i=0}^{\infty} 2^{-i\tilde{d}} Y_{v,2}^P =: \delta_j^P, \end{aligned}$$

where $P_i := Q(c_P, 2^{i-j_P+c_0})$ with $c_0 \in \mathbb{N}$, d is chosen such that $L + n - \frac{n}{r} + s_- - \frac{d}{q_+} > 0$ and $\tilde{d} \in (0, \infty)$.

From the above claim, (5.26), (2.1) and Remark 2.9(i), we deduce that

$$\sum_{j=(j_P \vee 0)}^{\infty} Y_j^P \lesssim 1 + \sum_{v=(j_P \vee 0)}^{\infty} \sum_{j=(j_P \vee 0)}^v 2^{(j-v)d} \sum_{i=0}^{\infty} 2^{-i\tilde{d}} Y_{v,2}^P \lesssim 1 + \sum_{i=0}^{\infty} 2^{-i\tilde{d}} \sum_{v=(j_P \vee 0)}^{\infty} Y_{v,2}^P \lesssim 1,$$

which implies that $I_{P,3} \lesssim 1$ and $\delta_j^P \in [2^{-j}, 2^{-j} + \theta]$ for some $\theta \in [0, \infty)$.

Therefore, to complete the estimate for $I_{P,3}$, it remains to prove the above claim (5.31). To this end, it suffices to show that, for all $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

$$\inf \left\{ \tilde{\lambda}_j \in (0, \infty) : \varrho_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})} \left(\frac{\chi_P [2^{js(\cdot)} \sum_{v=j}^{\infty} \sum_{\ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q|]^r}{[\phi(P)(\delta_j^P \tilde{\lambda}_j)^{1/q(\cdot)}]^r} \right) \leq 1 \right\} \lesssim 1,$$

which follows from the following estimate

$$(5.32) \quad H_j^P := \left\| \frac{\chi_P 2^{js(\cdot)r}}{\{\phi(P)[\delta_j^P]^{1/q(\cdot)}\}^r} \sum_{v=j}^{\infty} \sum_{\ell(Q)=2^{-v}} |t_Q|^r |\varphi_j * a_Q|^r \right\|_{L^{\frac{p(\cdot)}{r}}} \lesssim 1.$$

Next we show (5.32). By Lemma 5.11 and Remark 2.7, we find that

$$(5.33) \quad H_j^P \lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n)r} \left\| \frac{\chi_P 2^{js(\cdot)r}}{[\{\phi(P)[\delta_j^P]^{1/q(\cdot)}\}^r] \sum_{k \in \mathbb{Z}^n} \frac{|t_{Q_{vk}}|^r |Q_{vk}|^{-\frac{r}{2}}}{(1+2^j|\cdot-2^{-v}k|)^{Rr}}} \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)},$$

where R can be large enough. For all $x \in P$ and $v \in \mathbb{Z}_+$ with $v \geq j$, let

$$\Omega_{0,j}^{x,v} := \{k \in \mathbb{Z}^n : 2^j|x-2^{-v}k| \leq 1\}$$

and, for all $i \in \mathbb{N}$, $\Omega_{i,j}^{x,v} := \{k \in \mathbb{Z}^n : 2^{i-1} < 2^j|x-2^{-v}k| \leq 2^i\}$. Then, we see that, for all $x \in P$,

$$(5.34) \quad \begin{aligned} J(v, j, x, P) &:= \frac{2^{js(x)r}}{(\delta_j^P)^{\frac{r}{q(x)}}} \sum_{k \in \mathbb{Z}^n} \frac{|t_{Q_{vk}}|^r |Q_{vk}|^{-\frac{r}{2}}}{(1+2^j|x-2^{-v}k|)^{Rr}} \\ &\sim \frac{2^{js(x)r}}{(\delta_j^P)^{\frac{r}{q(x)}}} \sum_{i=0}^{\infty} \sum_{k \in \Omega_{i,j}^{x,v}} |t_{Q_{vk}}|^r |Q_{vk}|^{-\frac{r}{2}} 2^{-iRr} \\ &\sim \frac{2^{js(x)r}}{(\delta_j^P)^{\frac{r}{q(x)}}} \sum_{i=0}^{\infty} 2^{-iRr} 2^{vn} \int_{\cup_{\tilde{k} \in \Omega_{i,j}^{x,v}} Q_{v\tilde{k}}} \left[\sum_{k \in \Omega_{i,j}^{x,v}} |t_{Q_{vk}}| \tilde{\chi}_{Q_{vk}}(y) \right]^r dy. \end{aligned}$$

Since, for all $i \in \mathbb{Z}_+$, $v \in \mathbb{Z}_+$ with $v \geq j$, $x \in P$ and $y \in \cup_{\tilde{k} \in \Omega_{i,j}^{x,v}} Q_{v\tilde{k}}$, there exists $\tilde{k}_y \in \Omega_{i,j}^{x,v}$ such that $y \in Q_{v\tilde{k}_y}$, it follows that

$$(5.35) \quad 1 + 2^j|x-y| \leq 1 + 2^j|x-x_{Q_{v\tilde{k}_y}}| + 2^j|y-x_{Q_{v\tilde{k}_y}}| \lesssim 2^i + 2^{j-v} \lesssim 2^i$$

and hence

$$(5.36) \quad |y - c_P| \leq |y - x_{Q_{v\tilde{k}_y}}| + |x - x_{Q_{v\tilde{k}_y}}| + |x - c_P| \lesssim 2^{-v} + 2^{i-j} + 2^{-jP} \lesssim 2^{i-jP}.$$

By (5.36), we see that, for all $i \in \mathbb{Z}_+$, $v \in \mathbb{Z}_+$ with $v \geq j$ and $x \in P$,

$$\bigcup_{\tilde{k} \in \Omega_{i,j}^{x,v}} Q_{v\tilde{k}} \subset Q(c_P, 2^{i-jP+c_0}) =: Q_i^0$$

for some constant $c_0 \in \mathbb{N}$, which, combined with (5.34), (5.35) and Remark 3.11, implies that

$$(5.37) \quad \begin{aligned} J(v, j, x, P) &\lesssim (\delta_j^P)^{-\frac{r}{q(x)}} 2^{js(x)r} \sum_{i=0}^{\infty} 2^{-iRr} 2^{(v-j)n} 2^{i(a+\varepsilon)r} \\ &\quad \times \eta_{j,ar+\varepsilon r} * \left(\left[\sum_{\tilde{k} \in \Omega_{i,j}^{x,v}} |t_{Q_{v\tilde{k}}}| \tilde{\chi}_{Q_{v\tilde{k}}} \chi_{Q_i^0} \right]^r \right) (x) \\ &\lesssim 2^{(v-j)(n-rs_-)} \sum_{i=0}^{\infty} 2^{-ir(R-a-\varepsilon)} \eta_{j,ar} * \left(\left[\sum_{k \in \mathbb{Z}^n} (\delta_j^P)^{-\frac{1}{q(\cdot)}} \right. \right. \\ &\quad \left. \left. \times |t_{Q_{vk}}| 2^{vs(\cdot)} \tilde{\chi}_{Q_{vk}} \chi_{Q_i^0} \right]^r \right) (x), \end{aligned}$$

where $\varepsilon \in [C_{\log}(s) + C_{\log}(1/q), \infty)$. From this, (5.33), Lemma 3.12 and Remark 2.7(ii), we deduce that

$$H_j^P \lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n-\frac{n}{r}+s_-)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon)r} \frac{[\phi(Q_i^0)]^r}{[\phi(P)]^r}$$

$$\begin{aligned}
& \times \left\| \frac{(\delta_j^P)^{-r/q(\cdot)}}{\phi(Q_i^0)} \sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}| 2^{vs(\cdot)} \tilde{\chi}_{Q_{vk}} \right\|_{L^{p(\cdot)}(Q_i^0)}^r \\
& \lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n-\frac{n}{r}+s_--\frac{d}{q_+})r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon-\log c_1-\tilde{d}/q_-)r} \\
& \quad \times \left\| \frac{2^{(v-j)dr/q(\cdot)} 2^{-i\tilde{d}r/q(\cdot)}}{[\phi(Q_i^0)]^r (\delta_j^P)^{r/q(\cdot)}} \left[\sum_{k \in \mathbb{Z}^n} |t_{Q_{vk}}| 2^{vs(\cdot)} \tilde{\chi}_{Q_{vk}} \right] \right\|_{L^{\frac{p(\cdot)}{r}}(Q_i^0)}^r \\
& \lesssim \sum_{v=j}^{\infty} 2^{-(v-j)(L+n-\frac{n}{r}+s_--\frac{d}{q_+})r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon-\log c_1-\frac{\tilde{d}}{q_-})r} \lesssim 1,
\end{aligned}$$

where R is chosen large enough such that $R > a + \varepsilon + \log c_1 + \tilde{d}/q_-$, which completes the proof of that $I_{P,3} \lesssim 1$ and hence the case I.

Case II $q_+ = \infty$.

In this case, by Remark 2.10(iii), we see that $q(x) = \infty$ for all $x \in \mathbb{R}^n$. Thus, by Remark 2.9(v), we see that

$$\|t\|_{b_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)} = \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \in \mathbb{Z}_+, j \geq (j_P \vee 0)} \left\| \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-j}} |Q|^{-\frac{s(\cdot)}{n}} |t_Q| \tilde{\chi}_Q \right\|_{L^{p(\cdot)}(P)}.$$

Let P be a given dyadic cube. Then, by (5.24), we find that, for all $j \in \mathbb{Z}_+ \cap [j_P \vee 0, \infty)$,

$$\begin{aligned}
(5.38) \quad G_P^j &:= \frac{1}{\phi(P)} \left\| 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(P)} \\
&\lesssim \frac{1}{\phi(P)} \left\| 2^{js(\cdot)} \sum_{v=0}^{j-1} \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q| \right\|_{L^{p(\cdot)}(P)} \\
&\quad + \frac{1}{\phi(P)} \left\| 2^{js(\cdot)} \sum_{v=j}^{\infty} \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-v}} |t_Q| |\varphi_j * a_Q| \right\|_{L^{p(\cdot)}(P)} =: G_{P,1}^j + G_{P,2}^j.
\end{aligned}$$

To estimate $G_{P,1}^j$ and $G_{P,2}^j$, we let $\varepsilon \in (C_{\log}(s), \infty)$, $r \in (0, \min\{1, p_-\})$ and $a \in (n/r, \infty)$. For $G_{P,1}^j$, by an argument similar to that used in the estimate for $I_{P,1}$, we conclude that there exists a positive constant c_0 such that

$$\begin{aligned}
G_{P,1}^j &\lesssim \left\{ \sum_{v=0}^j 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\varepsilon/r)} \right. \\
&\quad \times \left. \frac{1}{\phi(P)} \left\| \sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-v}} |t_Q|^r 2^{vs(\cdot)r} |Q|^{-\frac{r}{2}} \chi_Q \right\|_{L^{\frac{p(\cdot)}{r}}(Q(c_P, 2^{i-v+c_0}))} \right\}^{\frac{1}{r}},
\end{aligned}$$

which, together with Remark 3.2, (2.4) and the facts that $c_1 \in [1, \infty)$ and $j \geq j_P$, implies that

$$(5.39) \quad G_{P,1}^j \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{v=0}^j 2^{(v-j)(K-s_+)r} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r})r} \frac{[\phi(Q(c_P, 2^{i-v}))]^r}{[\phi(P)]^r} \right\}^{\frac{1}{r}}$$

$$\begin{aligned}
&\lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{v=0}^j 2^{v(K-s_+-\log_2 c_1)} \sum_{i=0}^{\infty} 2^{-i(M-a-\frac{\varepsilon}{r}-\log_2 c_1)} 2^{j_P \log_2 c_1} \right\}^{\frac{1}{r}} \\
&\lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} 2^{-j \log_2 c_1} 2^{j_P \log_2 c_1} \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)}.
\end{aligned}$$

For $G_{P,2}^j$, by an argument similar to that used in the proof of (5.37), we find that there exists $c_0 \in \mathbb{N}$ such that

$$\begin{aligned}
G_{P,2}^j &\lesssim \frac{1}{\phi(P)} \left\{ \sum_{v=j+1}^{\infty} 2^{-(v-j)(L+n)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon)r} \right. \\
&\quad \times \left\| \eta_{j,ar} * \left(\left[\sum_{\substack{Q \in \mathcal{Q}^* \\ \ell(Q)=2^{-v}}} |t_Q| |Q|^{-\frac{s(\cdot)}{n}} \tilde{\chi}_Q \chi_{Q(c_P, 2^{i-j_P+c_0})} \right] \right)^r \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}^r \Bigg\}^{\frac{1}{r}},
\end{aligned}$$

which, combined with (2.4), Remarks 2.10(i) and 3.2, implies that

$$\begin{aligned}
G_{P,2}^j &\lesssim \frac{1}{\phi(P)} \left\{ \sum_{v=j+1}^{\infty} 2^{-(v-j)(L+n)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon)r} \right. \\
&\quad \times \left\| \left(\sum_{Q \in \mathcal{Q}^*, \ell(Q)=2^{-v}} |t_Q| |Q|^{-\frac{s(\cdot)}{n}} \tilde{\chi}_Q \right) \right\|_{L^{p(\cdot)}(Q(c_P, 2^{i-j_P+c_0}))}^r \Bigg\}^{\frac{1}{r}} \\
&\lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)} \left\{ \sum_{v=j+1}^{\infty} 2^{-(v-j)(L+n)r} \sum_{i=0}^{\infty} 2^{-i(R-a-\varepsilon-\log_2 c_1)r} \right\}^{\frac{1}{r}} \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)}.
\end{aligned}$$

By this, (5.38) and (5.39), we conclude that

$$\begin{aligned}
\|f\|_{B_{p(\cdot),\infty}^{s(\cdot),\phi}(\mathbb{R}^n)} &\lesssim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \in \mathbb{Z}_+ \cap [(j_P \vee 0), \infty)} \left\| 2^{js(\cdot)} |\varphi_j * f| \right\|_{L^{p(\cdot)}(P)} \\
&\lesssim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \sup_{j \in \mathbb{Z}_+ \cap [(j_P \vee 0), \infty)} (G_{P,1}^j + G_{P,2}^j) \lesssim \|t\|_{b_{p(\cdot),\infty}^{q(\cdot),\phi}(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of the case II.

Combining Cases I and II, we conclude that (5.25) holds true. This finishes the proof of Theorem 5.9. \square

Remark 5.13. We point out that the method used in the proof of Lemma 5.4 plays a very important role in the proof of Theorem 5.9. Precisely, the argument used in proofs of (5.31) and (5.29) is similar to that used in the proof of Lemma 5.4.

6 An application to trace operators

The purpose of this section is to study the trace of Besov-type spaces with variable smoothness and integrability.

Let $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$. Then, by Theorem 5.9, we can write $f = \sum_{Q \in \mathcal{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_Q\}_{Q \in \mathcal{Q}^*}$ is a family of smooth atoms of $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ and $\{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ satisfies

$$\|\{t_Q\}_{Q \in \mathcal{Q}^*}\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \leq C \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}$$

with C being a positive constant independent of f . Define the *trace* of f by setting, for all $\tilde{x} \in \mathbb{R}^{n-1}$,

$$(6.1) \quad \text{Tr}(f)(\tilde{x}) := \sum_{Q \in \mathcal{Q}^*} t_Q a_Q(\tilde{x}, 0).$$

This definition of $\text{Tr}(f)$ is determined canonical for all $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, since the actual construction of a_Q in the proof of Theorem 5.9 implies that $t_Q a_Q$ is obtained canonical. Moreover, Lemma 6.3 below shows that the summation in (6.1) converges in $\mathcal{S}'(\mathbb{R}^{n-1})$. Thus, the trace operator is well defined on $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$.

To state our main result of this section, we adopt the following notation. For p , q , s and ϕ as in Definition 2.12, let, for all $\tilde{x} \in \mathbb{R}^{n-1}$,

$$\tilde{p}(\tilde{x}) := p(\tilde{x}, 0), \quad \tilde{q}(\tilde{x}) := q(\tilde{x}, 0), \quad \tilde{s}(\tilde{x}) := s(\tilde{x}, 0)$$

and, for all cubes \tilde{Q} of \mathbb{R}^{n-1} , $\tilde{\phi}(\tilde{Q}) := \phi(\tilde{Q} \times [0, \ell(\tilde{Q})])$. In what follows, let $\mathbb{R}_+^n := \mathbb{R}^{n-1} \times [0, \infty)$, $\mathbb{R}_-^n := \mathbb{R}^{n-1} \times (-\infty, 0]$ and $k := (k_1, \dots, k_n) \in \mathbb{Z}^n$. Denote by $C_c^\infty(\mathbb{R})$ the set of all continuous functions f on \mathbb{R} with compact support satisfying that all classical derivatives of f are also continuous.

Theorem 6.1. *Let $n \geq 2$, $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$, $p, q \in C^{\log}(\mathbb{R}^n)$ and $s \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy*

$$(6.2) \quad s_- - \frac{1}{p_-} - (n-1) \left(\frac{1}{\min\{1, p_-\}} - 1 \right) > 0.$$

Then

$$\text{Tr } B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n) = B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}, \tilde{\phi}}(\mathbb{R}^{n-1}).$$

Remark 6.2. (i) Using quarkonial characterizations of both $B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n)$ and $B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$, Noi [51, Theorem 5.1] proved the following conclusion: $\text{Tr } B_{p(\cdot), q(\cdot)}^{s(\cdot)}(\mathbb{R}^n) = B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}}(\mathbb{R}^{n-1})$ under a weaker condition that

$$\text{ess inf}_{x \in \mathbb{R}^n} \left\{ s(x) - \frac{1}{p(x)} - (n-1) \left(\frac{1}{\min\{1, p(x)\}} - 1 \right) \right\} > 0,$$

but $s \in C^{\log}(\mathbb{R}^n)$ is required, which is stronger than the corresponding one in Theorem 6.1.

(ii) When $p_+ \in (0, \infty)$ and $q(\cdot) \equiv q \in (0, \infty)$ is a constant, the conclusion of Theorem 6.1 was proved by Moura et al. [44, Theorem 3.4] under the condition (6.2).

(iii) When p, q, s and ϕ are as in Remark 2.13(ii), Theorem 6.1 coincides with [78, Theorem 6.8].

The following conclusion implies that the summation in (6.1) converges in $\mathcal{S}'(\mathbb{R}^{n-1})$, whose proof is similar to that of [76, Lemma 4.3], the details being omitted.

Lemma 6.3. *Let n, p, q, s and ϕ be as in Theorem 6.1. Then, for all $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, $\text{Tr}(f) \in \mathcal{S}'(\mathbb{R}^{n-1})$.*

By Lemma 3.12 and an argument similar to that used in the proof of [76, Proposition 4.6], we obtain Lemma 6.4 below, the details being omitted. The corresponding result in the case that $\phi \equiv 1$ was obtained in [51, Lemma 5.3].

Lemma 6.4. *Let $p, q \in C^{\log}(\mathbb{R}^n)$, $s \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$. Let $\delta \in (0, \infty)$ and $\{E_Q\}_{Q \in \mathcal{Q}^*}$ be a collection of sets such that, for all $Q \in \mathcal{Q}^*$, $E_Q \subset 4Q$ and $|E_Q| \geq \delta|Q|$. Then, for all $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$, $t \in b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ if and only if*

$$\|t\|_{\widetilde{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{\substack{Q \in \mathcal{Q}^* \\ \ell(Q) = 2^{-j}}} 2^{j[s(\cdot) + \frac{n}{2}]} |t_Q| \chi_{E_Q} \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell_{q(\cdot)}(L^{p(\cdot)}(P))} < \infty.$$

In what follows, for all $j \in \mathbb{Z}$ and $\tilde{k} \in \mathbb{Z}^{n-1}$, let $\tilde{Q}_{j\tilde{k}} := 2^{-j}([0, 1)^{n-1} + \tilde{k})$ be the dyadic cube of \mathbb{R}^{n-1} , $\tilde{\mathcal{Q}}$ the set of all dyadic cubes of \mathbb{R}^{n-1} and $\tilde{\mathcal{Q}}^* := \{\tilde{Q} \in \tilde{\mathcal{Q}} : \ell(\tilde{Q}) \leq 1\}$. For all $\tilde{Q} \in \tilde{\mathcal{Q}}^*$ and $i \in \mathbb{Z}$, let $\tilde{\chi}_{\tilde{Q}} := |\tilde{Q}|^{-1/2} \chi_{\tilde{Q}}$, $I_{\tilde{Q}}^i := [(i-1)\ell(\tilde{Q}), i\ell(\tilde{Q}))$ and $(\hat{\tilde{Q}})_i := \tilde{Q} \times I_{\tilde{Q}}^i$. For all $P \in \mathcal{Q}$, denote by $P_{\mathbb{R}^{n-1}}^\perp$ the *vertical projection* of P on \mathbb{R}^{n-1} , namely,

$$P_{\mathbb{R}^{n-1}}^\perp := \{\tilde{x} \in \mathbb{R}^{n-1} : \exists x_n \in \mathbb{R} \text{ s. t. } (\tilde{x}, x_n) \in P\}$$

and, for all $j \in \mathbb{Z}_+$, let $P_{\mathbb{R}^{n-1}}^{\perp, j} := \{\tilde{Q} \in \tilde{\mathcal{Q}}^* : \tilde{Q} \subset P_{\mathbb{R}^{n-1}}^\perp, \ell(\tilde{Q}) = 2^{-j}\}$.

Applying Lemma 6.4, we conclude that the following conclusion holds true, which, in the case that $\phi \equiv 1$, was proved in [51, Lemma 5.4].

Lemma 6.5. *Let $p_1, p_2, q_1, q_2 \in C^{\log}(\mathbb{R}^n)$, $s_1, s_2 \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$. Assume that $p_1 = p_2$, $q_1 = q_2$ and $s_1 = s_2$ on \mathbb{R}_-^n or \mathbb{R}_+^n . Then, for all $\{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ and $i \in \{0, 1, 2\}$,*

$$\left\| \left\{ t_{(\hat{\tilde{Q}})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p_1(\cdot), q_1(\cdot)}^{s_1(\cdot), \phi}(\mathbb{R}^n)} \sim \left\| \left\{ t_{(\hat{\tilde{Q}})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p_2(\cdot), q_2(\cdot)}^{s_2(\cdot), \phi}(\mathbb{R}^n)},$$

where the implicit positive constants are independent of $\{t_Q\}_{Q \in \mathcal{Q}^*}$.

Proof. By similarity, we only consider the case that $p_1 = p_2$, $q_1 = q_2$ and $s_1 = s_2$ on \mathbb{R}_+^n . For all $\tilde{Q} \in \tilde{\mathcal{Q}}^*$ and $i \in \{0, 1, 2\}$, let

$$E_{(\hat{\tilde{Q}})_i} := \left\{ (\tilde{x}, x_n) \in \mathbb{R}^n : \tilde{x} \in \tilde{Q}, \frac{i+1}{2}\ell(\tilde{Q}) \leq x_n < \frac{3(i+1)}{4}\ell(\tilde{Q}) \right\}.$$

Then $E_{(\hat{\tilde{Q}})_i} \subset \mathbb{R}_+^n$, $E_{(\hat{\tilde{Q}})_i} \subset 4(\hat{\tilde{Q}})_i$ and $|E_{(\hat{\tilde{Q}})_i}| \geq \frac{4}{i+1}|(\hat{\tilde{Q}})_i|$. By this and Lemma 6.4, we conclude that

$$\begin{aligned} & \left\| \left\{ t_{(\hat{\tilde{Q}})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p_1(\cdot), q_1(\cdot)}^{s_1(\cdot), \phi}(\mathbb{R}^n)} \\ & \sim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js_1(\cdot)} \sum_{\tilde{Q} \in P_{\mathbb{R}^{n-1}}^{\perp, j}} |t_{(\hat{\tilde{Q}})_i}| |(\hat{\tilde{Q}})_i|^{-\frac{1}{2}} \chi_{E_{(\hat{\tilde{Q}})_i}} \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell_{q_1(\cdot)}(L^{p_1(\cdot)}(P))} \end{aligned}$$

$$\begin{aligned}
& \sim \sup_{P \in \mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js_2(\cdot)} \sum_{\tilde{Q} \in P_{\mathbb{R}^{n-1}}^{\perp, j}} |t_{(\tilde{Q})_i}| |(\tilde{Q})_i|^{-\frac{1}{2}} \chi_{E_{(\tilde{Q})_i}} \right\}_{j \geq (j_P \vee 0)} \right\|_{\ell^{q_2(\cdot)}(L^{p_2(\cdot)}(P))} \\
& \sim \left\| \left\{ t_{(\tilde{Q})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p_2(\cdot), q_2(\cdot)}^{s_2(\cdot), \phi}(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of Lemma 6.5. \square

Adopting an argument similar to that used in the proof [16, Proposition 7.3], we obtain the following conclusion, the details being omitted.

Corollary 6.6. *Let $p_1, p_2, q_1, q_2 \in C^{\log}(\mathbb{R}^n)$, $s_1, s_2 \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$. Assume that $p_1 = p_2, q_1 = q_2$ and $s_1 = s_2$ on $\mathbb{R}^{n-1} \times \{0\}$. Then, for all $t := \{t_{(\tilde{Q})_i}\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \subset \mathbb{C}$ and $i \in \{0, 1, 2\}$,*

$$\left\| \left\{ t_{(\tilde{Q})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p_1(\cdot), q_1(\cdot)}^{s_1(\cdot), \phi}(\mathbb{R}^n)} \sim \left\| \left\{ t_{(\tilde{Q})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p_2(\cdot), q_2(\cdot)}^{s_2(\cdot), \phi}(\mathbb{R}^n)},$$

where the implicit positive constants are independent of t .

For the notation simplicity, let $\tilde{\beta}(\tilde{x}) := \tilde{s}(\tilde{x}) - \frac{1}{\tilde{p}(\tilde{x})}$ for all $\tilde{x} \in \mathbb{R}^{n-1}$.

Lemma 6.7. *Let $p, q \in C^{\log}(\mathbb{R}^n)$, $s \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}_+^{n+1})$. Then there exists a positive constant C such that, for all $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ and $i \in \{0, 1, 2\}$,*

$$(6.3) \quad \left\| \left\{ t_{(\tilde{Q})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{\tilde{p}(\cdot)}, \tilde{\phi}}(\mathbb{R}^{n-1})} \sim \left\| \left\{ t_{(\tilde{Q})_i} [\ell(\tilde{Q})]^{\frac{1}{2}} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)},$$

where the implicit positive constants are independent of t .

Proof. By similarity, we only give the proof of “ \lesssim ” in (6.3). By Corollary 6.6, we may assume that p, q and s are independent of the n -th coordinate x_n with $|x_n| \leq 2$. For all $\tilde{P} \in \tilde{\mathcal{Q}}$, $j \in \mathbb{Z}_+$, $\tilde{x} \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}^n$, let $\tilde{P} := \tilde{P} \times [0, \ell(\tilde{P}))$,

$$\Gamma_{\tilde{P}}^j := \left\{ Q \in \tilde{\mathcal{Q}}^* : \tilde{Q} \subset \tilde{P}, \ell(\tilde{Q}) = 2^{-j} \right\}, \quad H_{\tilde{P}}^j(\tilde{x}) := \sum_{\tilde{Q} \in \Gamma_{\tilde{P}}^j} |t_{(\tilde{Q})_i}| \tilde{\chi}_{\tilde{Q}}(\tilde{x})$$

and

$$G_{\tilde{P}}^j(x) := \sum_{\tilde{Q} \in \Gamma_{\tilde{P}}^j} |t_{(\tilde{Q})_i}| [\ell(\tilde{Q})]^{\frac{1}{2}} \tilde{\chi}_{(\tilde{Q})_i}(x).$$

Let $\tilde{P} \in \tilde{\mathcal{Q}}$ be a given dyadic cube. Then, by (2.4), we find that, for all $j \in \mathbb{Z}_+ \cap [(j_{\tilde{P}} \vee 0), \infty)$ and $\lambda, \mu \in (0, \infty)$,

$$\begin{aligned}
& \int_{\tilde{P}} \left[\frac{1}{\mu} \left\{ [\lambda \tilde{\phi}(\tilde{P})]^{-1} 2^{j\tilde{\beta}(\tilde{x})} H_{\tilde{P}}^j(\tilde{x}) \right\}^{\tilde{q}(\tilde{x})} \right]^{\frac{\tilde{p}(\tilde{x})}{\tilde{q}(\tilde{x})}} d\tilde{x} \\
& = \int_{\tilde{P}} \left[\int_{(i-1)2^{-j}}^{i2^{-j}} \frac{2^{j\tilde{s}(\tilde{x})\tilde{p}(\tilde{x})}}{\mu^{\tilde{p}(\tilde{x})/\tilde{q}(\tilde{x})}} \left\{ [\lambda \tilde{\phi}(\tilde{P})]^{-1} H_{\tilde{P}}^j(\tilde{x}) \right\}^{\tilde{p}(\tilde{x})} \chi_{I_{\tilde{Q}}^i}(x_n) dx_n \right] d\tilde{x}
\end{aligned}$$

$$\begin{aligned}
&= \int_{\tilde{P}} \int_{(i-1)2^{-j}}^{i2^{-j}} \frac{2^{js(\tilde{x}, x_n)} p(\tilde{x}, x_n)}{\mu^{p(\tilde{x}, x_n)/q(\tilde{x}, x_n)}} \left\{ [\lambda \tilde{\phi}(\tilde{P})]^{-1} H_{\tilde{P}}^j(\tilde{x}) \chi_{I_{\tilde{Q}}^i}(x_n) \right\}^{p(\tilde{x}, x_n)} dx_n d\tilde{x} \\
&\lesssim \int_{4\hat{\tilde{P}}} \left[\frac{1}{\mu} \left\{ \left[\lambda \phi(4\hat{\tilde{P}}) \right]^{-1} 2^{js(x)} G_{\tilde{P}}^j(x) \right\}^{q(x)} \right]^{\frac{p(x)}{q(x)}} dx,
\end{aligned}$$

which implies that

$$\left\| \left\{ [\lambda \phi(\tilde{P})]^{-1} 2^{j\tilde{\beta}(\cdot)} H_{\tilde{P}}^j \right\}_{j \geq (j_{\tilde{P}} \vee 0)}^{\tilde{q}(\cdot)} \right\|_{L^{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}(\tilde{P})} \lesssim \left\| \left\{ \left[\lambda \phi(4\hat{\tilde{P}}) \right]^{-1} 2^{js(\cdot)} G_{\tilde{P}}^j \right\}^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}(4\hat{\tilde{P}})}.$$

From this and Remark 3.2, we deduce that

$$\begin{aligned}
&\frac{1}{\tilde{\phi}(\tilde{P})} \left\| \left\{ 2^{j\tilde{\beta}(\cdot)} H_{\tilde{P}}^j \right\}_{j \geq (j_{\tilde{P}} \vee 0)} \right\|_{\ell^{\tilde{q}(\cdot)}(L^{\tilde{p}(\cdot)}(\tilde{P}))} \\
&= \inf \left\{ \lambda \in (0, \infty) : \sum_{j=(j_{\tilde{P}} \vee 0)}^{\infty} \left\| \left\{ [\lambda \tilde{\phi}(\tilde{P})]^{-1} 2^{j\tilde{\beta}(\cdot)} H_{\tilde{P}}^j \right\}^{\tilde{q}(\cdot)} \right\|_{L^{\frac{\tilde{p}(\cdot)}{\tilde{q}(\cdot)}}(\tilde{P})} \leq 1 \right\} \\
&\lesssim \frac{1}{\phi(4\hat{\tilde{P}})} \left\| \left\{ 2^{js(\cdot)} G_{\tilde{P}}^j \right\}_{j \geq (j_{\tilde{P}} \vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(4\hat{\tilde{P}}))} \lesssim \left\| \left\{ t_{(\tilde{Q})_i} [\ell(\tilde{Q})]^{-\frac{1}{2}} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)},
\end{aligned}$$

which, combined with the arbitrariness of \tilde{P} , further implies that

$$\left\| \left\{ t_{(\tilde{Q})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot) - \frac{1}{p(\cdot)}, \tilde{\phi}}(\mathbb{R}^{n-1})} \lesssim \left\| \left\{ t_{(\tilde{Q})_i} [\ell(\tilde{Q})]^{\frac{1}{2}} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

This finishes the proof of Lemma 6.7. \square

Proof of Theorem 6.1. Let $f \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$. Then, by Theorem 5.9, we have an atomic decomposition $f = \sum_{Q \in \mathcal{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_Q\}_{Q \in \mathcal{Q}^*}$ is a family of smooth atoms of $B_{p(\cdot), p(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ and $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ satisfies

$$(6.4) \quad \|t\|_{f_{p(\cdot), p(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim \|f\|_{F_{p(\cdot), p(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

Since $\text{supp } a_Q \subset 3Q$ for each $Q \in \mathcal{Q}^*$, it follows that, if $i \notin \{0, 1, 2\}$, then, for each $\tilde{Q} \in \tilde{\mathcal{Q}}^*$, $a_{(\tilde{Q})_i}(\cdot, 0) = 0$, which implies that $\text{Tr}(f)$ can be rewritten as, for all $\tilde{x} \in \mathbb{R}^{n-1}$,

$$(6.5) \quad \text{Tr}(f)(\tilde{x}, 0) = \sum_{i=0}^2 \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} t_{(\tilde{Q})_i} a_{(\tilde{Q})_i}(\tilde{x}, 0) =: \sum_{i=0}^2 \sum_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \lambda_{(\tilde{Q})_i} b_{(\tilde{Q})_i}(\tilde{x}),$$

where, for each $\tilde{Q} \in \tilde{\mathcal{Q}}^*$ and $\tilde{x} \in \mathbb{R}^{n-1}$, $b_{(\tilde{Q})_i}(\tilde{x}) := [\ell(\tilde{Q})]^{\frac{1}{2}} a_{(\tilde{Q})_i}(\tilde{x}, 0)$ and $\lambda_{(\tilde{Q})_i} := [\ell(\tilde{Q})]^{-\frac{1}{2}} t_{(\tilde{Q})_i}$.

Since a_Q is a smooth atom supported near Q of $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$, by (6.2), we easily find that, for each $\tilde{Q} \in \tilde{\mathcal{Q}}^*$, $b_{(\tilde{Q})_i}$ is also a smooth atom of $B_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot), \tilde{\phi}}(\mathbb{R}^{n-1})$ supported near \tilde{Q} . On the other hand, by Lemma 6.7 and (6.4), we find that

$$\left\| \left\{ \lambda_{(\tilde{Q})_i} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{\tilde{p}(\cdot), \tilde{q}(\cdot)}^{\tilde{s}(\cdot), \tilde{\phi}}(\mathbb{R}^{n-1})} \lesssim \left\| \left\{ \lambda_{(\tilde{Q})_i} [\ell(\tilde{Q})]^{\frac{1}{2}} \right\}_{\tilde{Q} \in \tilde{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}$$

$$\sim \left\| \left\{ t_{(\widehat{Q})_i} \right\}_{\widehat{Q} \in \widehat{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

Therefore, by Theorem 5.9 and (6.5), we conclude that

$$\|\mathrm{Tr}(f)\|_{B_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})} \lesssim \sum_{i=0}^2 \left\| \left\{ \lambda_{(\widehat{Q})_i} \right\}_{\widehat{Q} \in \widehat{\mathcal{Q}}^*} \right\|_{b_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)}.$$

Conversely, we prove that the operator Tr is surjective. Let $f \in B_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})$. Then, by Theorem 5.9, we find that there exist a sequence $\{\lambda_{\widehat{Q}}\}_{\widehat{Q} \in \widehat{\mathcal{Q}}^*} \subset \mathbb{C}$ and a family $\{a_{\widehat{Q}}\}_{\widehat{Q} \in \widehat{\mathcal{Q}}^*}$ of smooth atoms of $B_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})$ such that $f = \sum_{\widehat{Q} \in \widehat{\mathcal{Q}}^*} \lambda_{\widehat{Q}} a_{\widehat{Q}}$ in $\mathcal{S}'(\mathbb{R}^{n-1})$ and

$$(6.6) \quad \|\{\lambda_{\widehat{Q}}\}_{\widehat{Q} \in \widehat{\mathcal{Q}}^*}\|_{b_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})}.$$

Similar to the proof of [78, Theorem 6.8], we choose a function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $\mathrm{supp} \eta \subset (-1/2, 1/2)$ and $\eta(0) = 1$. For all $\widetilde{Q} \in \widetilde{\mathcal{Q}}^*$ and $\xi \in \mathbb{R}$, let $\eta_{\widetilde{Q}}(\xi) := \eta(2^{-\log_2 \ell(\widetilde{Q})} \xi)$. Then $\mathrm{supp} \eta_{\widetilde{Q}} \subset (-\ell(\widetilde{Q}), \ell(\widetilde{Q}))$. Let

$$(6.7) \quad g := \sum_{\widetilde{Q} \in \widetilde{\mathcal{Q}}^*} \lambda_{\widetilde{Q}} a_{\widetilde{Q}} \otimes \eta_{\widetilde{Q}} =: \sum_{i=0}^1 \sum_{\widetilde{Q} \in \widetilde{\mathcal{Q}}^*} t_{(\widehat{Q})_i} b_{(\widehat{Q})_i},$$

where, for all $Q \in \mathcal{Q}^*$ and $(\widetilde{x}, x_n) \in \mathbb{R}^n$,

$$b_Q(\widetilde{x}, x_n) := [\ell(\widetilde{Q})]^{-\frac{1}{2}} a_{\widetilde{Q}} \otimes \eta_{\widetilde{Q}}(\widetilde{x}, x_n) =: [\ell(\widetilde{Q})]^{-\frac{1}{2}} a_{\widetilde{Q}}(\widetilde{x}) \eta_{\widetilde{Q}}(x_n),$$

$t_Q := [\ell(\widetilde{Q})]^{1/2} \lambda_{\widetilde{Q}}$ if $Q = (\widehat{Q})_i$ for some $i \in \{0, 1\}$ and $t_Q := 0$ otherwise. By the construction of $\{b_Q\}_{Q \in \mathcal{Q}^*}$, we easily find that, for each $Q \in \mathcal{Q}^*$, b_Q is a smooth atom supported near Q of $B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$. On the other hand, by Lemma 6.7 and (6.6), we conclude that, for each $i \in \{0, 1\}$,

$$\begin{aligned} \left\| \left\{ t_{(\widehat{Q})_i} \right\}_{\widehat{Q} \in \widehat{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} &\sim \left\| \left\{ \lambda_{\widetilde{Q}} [\ell(\widetilde{Q})]^{1/2} \right\}_{\widetilde{Q} \in \widetilde{\mathcal{Q}}^*} \right\|_{b_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \\ &\lesssim \left\| \{\lambda_{\widetilde{Q}}\}_{\widetilde{Q} \in \widetilde{\mathcal{Q}}^*} \right\|_{b_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})}, \end{aligned}$$

which, together with Theorem 5.9, implies that the summation in (6.7) converges in $\mathcal{S}'(\mathbb{R}^n)$, $g \in B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)$ and $\|g\|_{B_{p(\cdot), q(\cdot)}^{s(\cdot), \phi}(\mathbb{R}^n)} \lesssim \|f\|_{B_{\widetilde{p}(\cdot), \widetilde{q}(\cdot)}^{\widetilde{\beta}(\cdot), \widetilde{\phi}}(\mathbb{R}^{n-1})}$; furthermore, $\mathrm{Tr}(g) = f$ in $\mathcal{S}'(\mathbb{R}^{n-1})$. Therefore, Tr is surjective. This finishes the proof of Theorem 6.1. \square

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